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On a symmetric identity of Ramanujan involving the sum of two squares function

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ABSTRACT

In his paper "*On the expression of a number as the sum of two squares*", G.H. Hardy recorded a beautiful identity of Ramanujan, as well as its generalization, and described briefly their proofs. In this article, we give full details of Hardy's (and very likely Ramanujan's) proofs and present some generalizations of Ramanujan's identity.

1. Introduction

Let $r_k(n)$ denote the number of representations of the non-negative integer n as a sum of k squares. The Gauss circle problem asks for the number of integer lattice points contained in a circle of radius m centered at the origin. A good approximation to the answer is given by the area of the circle πm^2. The circle problem can therefore be reformulated as finding a good estimate of the error term

$$E(m) = \left| \sum_{0\le n\le m^2} r_2(n) - \pi m^2 \right|$$

as $m \to \infty$. Hardy was one of the first few mathematicians who made significant contributions to the estimate of the error term $E(m)$. In his study of this problem [11], he discovered the identity

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$$\sum_{n=1}^{\infty} r_2(n)\exp\left(-s\sqrt{n}\right) = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2+4\pi^2 n)^{3/2}}. \tag{1}$$

Here and throughout, we use $\exp(x)$ to denote the exponential function. Hardy proved (1) using the transformation formula for Jacobi's theta function

$$\vartheta_3(\tau) = \sum_{k=-\infty}^{\infty} \exp\left(\pi i \tau k^2\right)$$

and the integral

$$\exp\left(-s\sqrt{\mu^2+\nu^2}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp\left(-\omega^2 - \frac{s^2(\mu^2+\nu^2)}{4\omega^2}\right)\, d\omega. \tag{2}$$

At the end of his paper [11, p. 283], Hardy gave the following statement: "Mr S. Ramanujan has indicated to me an interesting formula which contains (1) as a special case." The formula that Hardy referred to is the following:

Theorem 1. *For any positive real numbers α and β,*

$$\sum_{n=0}^{\infty} r_2(n) \frac{\exp\left(-2\pi\sqrt{(n+\alpha)\beta}\right)}{\sqrt{n+\alpha}} = \sum_{n=0}^{\infty} r_2(n) \frac{\exp\left(-2\pi\sqrt{(n+\beta)\alpha}\right)}{\sqrt{n+\beta}}. \tag{3}$$

The symmetry in α and β in (3) is particularly striking. On multiple occasions—including in New Zealand in 2004, in Singapore in 2017 and most recently in the USA in 2018—B.C. Berndt highlighted the "beauty" of (3). According to Berndt (see [2, p. 126]), this identity does not appear in any of Ramanujan's published papers nor in his notebooks.

Hardy showed that his identity (1) is a consequence of (3) by first differentiating both sides of (3) with respect to β, letting $\alpha \to 0$, and replacing $2\pi\sqrt{\beta}$ with s. Hardy then presented a generalization of (3), also due to Ramanujan, where the quantity $r_2(n)$ was replaced by the number of representations of n by a general positive definite binary quadratic form $ax^2 + bxy + cy^2$. Hardy concluded his article by stating that both (3) and its generalization may be deduced from "the formulae for the linear transformation of the double Theta-functions." In the next section, we fill in the details of the proof suggested by Hardy. We surmise that our approach is likely to be similar to that of Ramanujan. In Section 3, we further generalize (3) to one involving $2\ell+1$ copies of a quadratic form, as well as another generalization involving powers of the Dedekind eta function.

2. The proof of Theorem 1

There are two key results involved in the proof of (3). The first is the following integral evaluation [8, Formula 3.325]:

Theorem 2. *Let ξ and η be positive real numbers. Then*

$$\int_0^{\infty} \exp\left(-\xi t^2 - \frac{\eta}{t^2}\right) dt = \frac{1}{2}\sqrt{\frac{\pi}{\xi}} \exp\left(-2\sqrt{\xi\eta}\right).$$

Proof. We follow the proof given in [3, p. 172]. For real positive c, define the integral

$$I(c) = \int_0^\infty \exp\left(-s^2 - \frac{c}{s^2}\right)\, ds. \tag{4}$$

Let $s = \frac{\sqrt{c}}{\nu}$ and deduce that

$$I(c) = \int_0^\infty \exp\left(-\nu^2 - \frac{c}{\nu^2}\right) \frac{\sqrt{c}}{\nu^2}\, d\nu. \tag{5}$$

Replacing the variable ν in (5) by s and adding the resulting identity to (4), we find that

$$2I(c) = \int_0^\infty \exp\left(-s^2 - \frac{c}{s^2}\right)\left(1 + \frac{\sqrt{c}}{s^2}\right) ds.$$

Setting $\mu = s - \frac{\sqrt{c}}{s}$, we deduce that

$$2I(c) = \int_{-\infty}^\infty \exp\left(-\mu^2 - 2\sqrt{c}\right) d\mu = \exp\left(-2\sqrt{c}\right)\sqrt{\pi}, \tag{6}$$

since

$$\int_{-\infty}^\infty \exp\left(-\mu^2\right) d\mu = \sqrt{\pi}.$$

Now, consider the integral

$$J(\xi, \eta) = \int_0^\infty \exp\left(-\xi t^2 - \frac{\eta}{t^2}\right) dt.$$

Let $t = \frac{\omega}{\sqrt{\xi}}$. Then

$$J(\xi, \eta) = \frac{1}{\sqrt{\xi}} \int_0^\infty \exp\left(-\omega^2 - \frac{\xi\eta}{\omega^2}\right) d\omega = \frac{1}{\sqrt{\xi}} I(\xi\eta) = \frac{1}{2}\sqrt{\frac{\pi}{\xi}} \exp\left(-2\sqrt{\xi\eta}\right),$$

where we have used (6) in our last equality. □

The second result we require is the transformation formula for theta functions alluded to by Hardy. There are several versions of the transformation formula in the literature. The proof of the following version can be found in [12, pp. 204–205].

Theorem 3. *Let h, k be rational numbers, a, b, c be integers such that $a > 0$ and $D = 4ac - b^2 > 0$. Let $Q(x, y) = ax^2 + bxy + cy^2$. Then for any positive real number t,*

$$\sum_{(m,n)\in\mathbf{Z}^2} \exp\left(\frac{-2\pi t Q(m+h,n+k)}{\sqrt{D}}\right)$$
$$= \frac{1}{t} \sum_{(m,n)\in\mathbf{Z}^2} \exp\left(\frac{-2\pi Q(m,n)}{t\sqrt{D}}\right) \exp\left(2\pi i(mk-nh)\right).$$

We are now ready to establish the following:

Theorem 4. *Let $Q(x,y)$, D, h and k be defined as in Theorem 3. Then for any positive real numbers α and β,*

$$\sum_{(m,n)\in\mathbf{Z}^2} \frac{\exp\left(-4\pi\sqrt{\frac{(Q(m+h,n+k)+\alpha)\beta}{D}}\right)}{\sqrt{Q(m+h,n+k)+\alpha}}$$
$$= \sum_{(m,n)\in\mathbf{Z}^2} \frac{\exp\left(2\pi i(mk-nh)\right)\exp\left(-4\pi\sqrt{\frac{(Q(m,n)+\beta)\alpha}{D}}\right)}{\sqrt{Q(m,n)+\beta}}.$$

Proof. Let

$$\xi = \frac{2\pi}{\sqrt{D}}\left(Q(m+h,n+k)+\alpha\right) \text{ and } \eta = \frac{2\pi\beta}{\sqrt{D}}.$$

Using Theorem 2, we deduce that

$$\frac{D^{1/4}}{2\sqrt{2}} \frac{\exp\left(-4\pi\sqrt{\frac{(Q(m+h,n+k)+\alpha)\beta}{D}}\right)}{\sqrt{Q(m+h,n+k)+\alpha}}$$
$$= \int_0^\infty \exp\left(-\frac{2\pi t^2(Q(m+h,n+k)+\alpha)}{\sqrt{D}} - \frac{2\pi\beta}{t^2\sqrt{D}}\right) dt$$
$$= \int_0^\infty \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\alpha t^2 + \frac{\beta}{t^2}\right)\right) \exp\left(\frac{-2\pi t^2 Q(m+h,n+k)}{\sqrt{D}}\right) dt.$$

Hence

$$\frac{D^{1/4}}{2\sqrt{2}} \sum_{(m,n)\in\mathbf{Z}^2} \frac{\exp\left(-4\pi\sqrt{\frac{(Q(m+h,n+k)+\alpha)\beta}{D}}\right)}{\sqrt{Q(m+h,n+k)+\alpha}}$$
$$= \int_0^\infty \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\alpha t^2 + \frac{\beta}{t^2}\right)\right) \sum_{(m,n)\in\mathbf{Z}^2} \exp\left(\frac{-2\pi t^2 Q(m+h,n+k)}{\sqrt{D}}\right) dt$$
$$= \int_0^\infty \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\alpha t^2 + \frac{\beta}{t^2}\right)\right) \frac{1}{t^2} \sum_{(m,n)\in\mathbf{Z}^2} \exp\left(2\pi i(mk-nh)\right) \exp\left(\frac{-2\pi Q(m,n)}{t^2\sqrt{D}}\right) dt, \tag{7}$$

where we have used Theorem 3 in the last equality. Letting $s = \frac{1}{t}$, we deduce from (7) and Theorem 2 that

$$\begin{aligned}
&\int_0^\infty \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\alpha t^2+\frac{\beta}{t^2}\right)\right)\frac{1}{t^2}\sum_{(m,n)\in\mathbf{Z}^2}\exp\left(2\pi i(mk-nh)\right)\exp\left(\frac{-2\pi Q(m,n)}{t^2\sqrt{D}}\right)dt\\
&=\int_0^\infty \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\frac{\alpha}{s^2}+\beta s^2\right)\right)\sum_{(m,n)\in\mathbf{Z}^2}\exp\left(2\pi i(mk-nh)\right)\exp\left(\frac{-2\pi s^2 Q(m,n)}{\sqrt{D}}\right)ds\\
&=\sum_{(m,n)\in\mathbf{Z}^2}\exp\left(2\pi i(mk-nh)\right)\int_0^\infty \exp\left(-\frac{2\pi\alpha}{s^2\sqrt{D}}-\frac{2\pi s^2(Q(m,n)+\beta)}{\sqrt{D}}\right)ds\\
&=\frac{D^{1/4}}{2\sqrt{2}}\sum_{(m,n)\in\mathbf{Z}^2}\exp\left(2\pi i(mk-nh)\right)\frac{\exp\left(-4\pi\sqrt{\frac{(Q(m,n)+\beta)\alpha}{D}}\right)}{\sqrt{Q(m,n)+\beta}}. \qquad (8)
\end{aligned}$$

Equating the last expression of (8) with the first expression of (7), we conclude the proof of the theorem. □

Substituting $h = k = 0$, we obtain Ramanujan's generalization of (3) that was quoted by Hardy, namely,

$$\sum_{(m,n)\in\mathbf{Z}^2}\frac{\exp\left(-4\pi\sqrt{\frac{(Q(m,n)+\alpha)\beta}{D}}\right)}{\sqrt{Q(m,n)+\alpha}}=\sum_{(m,n)\in\mathbf{Z}^2}\frac{\exp\left(-4\pi\sqrt{\frac{(Q(m,n)+\beta)\alpha}{D}}\right)}{\sqrt{Q(m,n)+\beta}}.$$

Theorem 1 follows from setting $Q(m,n) = m^2 + n^2$ and rewriting the sum in terms of $r_2(n)$.

If we take $h = k = \frac{1}{2}$, and $Q(m,n) = m^2+n^2$ in Theorem 4, we have the following analogue of Theorem 1, where the generating function $\vartheta_3^2(\tau)$ in the left and right sides of Ramanujan's identity is replaced by $\vartheta_2^2(\tau)$ and $\vartheta_4^2(\tau)$ respectively.

Corollary 2.1. *For any positive real numbers α and β,*

$$\begin{aligned}
&\sum_{(m,n)\in\mathbf{Z}^2}\frac{\exp\left(-2\pi\sqrt{\left(\left(m+\frac12\right)^2+\left(n+\frac12\right)^2+\alpha\right)\beta}\right)}{\sqrt{\left(m+\frac12\right)^2+\left(n+\frac12\right)^2+\alpha}}\\
&\qquad=\sum_{(m,n)\in\mathbf{Z}^2}\frac{(-1)^{m+n}\exp\left(-2\pi\sqrt{(m^2+n^2+\beta)\alpha}\right)}{\sqrt{m^2+n^2+\beta}}.
\end{aligned}$$

In addition, by taking $h = k = \frac{1}{3}$ and $Q(m,n) = m^2 + mn + n^2$, we obtain another analogue of Ramanujan's identity that relates the cubic theta functions $c(q)$ and $b(q)$. (See [4] for the definitions and properties satisfied by these functions.)

Corollary 2.2. *For any positive real numbers α and β,*

$$\begin{aligned}
&\sum_{(m,n)\in\mathbf{Z}^2}\frac{\exp\left(-4\pi\sqrt{\frac{\left((m+\frac13)^2+(m+\frac13)(n+\frac13)+(n+\frac13)^2+\alpha\right)\beta}{3}}\right)}{\sqrt{\left(m+\frac13\right)^2+\left(m+\frac13\right)\left(n+\frac13\right)+\left(n+\frac13\right)^2+\alpha}}\\
&\qquad=\sum_{(m,n)\in\mathbf{Z}^2}\frac{\exp\left(\frac{2\pi i(m-n)}{3}\right)\exp\left(-4\pi\sqrt{\frac{(m^2+mn+n^2+\beta)\alpha}{3}}\right)}{\sqrt{m^2+mn+n^2+\beta}}.
\end{aligned}$$

3. Further generalizations of Ramanujan's identity (3)

Let $(v)_0 = 1$ and $(v)_k = (v)(v+1)\cdots(v+k-1)$ for $k \in \mathbf{Z}^+$. Define

$$ {}_2F_0\left(a, b; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!} z^k. $$

We are interested in the special case where either a or b is a negative integer so that the above hypergeometric series terminates into a polynomial. We now generalize Ramanujan's identity (3) to one involving $2\ell+1$ copies of $Q(x,y)$. With $Q(x,y)$ fixed, let $r_{Q,k}(n)$ denote the number of representations of n as a sum of k integers of the form $Q(x,y)$. Our generalization is as follows:

Theorem 5. *Let ℓ be a non-negative integer and let $Q(x,y)$ and D be defined as in Theorem 3. Then*

$$ \sum_{n\in\mathbf{Z}} r_{Q,2\ell+1}(n) \exp\left(-4\pi\sqrt{\frac{(n+\alpha)\beta}{D}}\right) \sqrt{\frac{\beta^\ell}{(n+\alpha)^{\ell+1}}} \cdot {}_2F_0\left(-\ell, \ell+1; \frac{-1}{8\pi\sqrt{\frac{(n+\alpha)\beta}{D}}}\right) $$
$$ = \sum_{n\in\mathbf{Z}} r_{Q,2\ell+1}(n) \exp\left(-4\pi\sqrt{\frac{(n+\beta)\alpha}{D}}\right) \sqrt{\frac{\alpha^\ell}{(n+\beta)^{\ell+1}}} \cdot {}_2F_0\left(-\ell, \ell+1; \frac{-1}{8\pi\sqrt{\frac{(n+\beta)\alpha}{D}}}\right). $$

We consider the case of Theorem 5 where $Q(x,y) = x^2+y^2$. We observe that when $\ell = 0$ the identity reduces to (3). When $\ell = 1$, ${}_2F_0\left(-1, 2; z\right) = 1 - 2z$. Since $r_{x^2+y^2,3}(n)$ is equivalent to the number of representations of n by a sum of six squares, we deduce the following corollary.

Corollary 3.1. *For any positive real numbers α and β,*

$$ \sum_{n=0}^{\infty} r_6(n) \exp\left(-2\pi\sqrt{(n+\alpha)\beta}\right) \frac{1+2\pi\sqrt{(n+\alpha)\beta}}{\sqrt{(n+\alpha)^3}} $$
$$ = \sum_{n=0}^{\infty} r_6(n) \exp\left(-2\pi\sqrt{(n+\beta)\alpha}\right) \frac{1+2\pi\sqrt{(n+\beta)\alpha}}{\sqrt{(n+\beta)^3}}. $$

For other values of ℓ, we obtain an identity symmetric in α and β for $r_k(n)$ when k is of the form $4\ell+2$. This is surprising since historically, identities associated with $r_k(n)$ with k of the form 4ℓ tend to be more elegant. (See for example [7].) Our identity involving $r_{4\ell+2}(n)$ is equivalent to [2, (8.17)] with k specialized to $4\ell+2$. The identity given in [2, (8.17)] is a consequence of [1, Theorem 9.1] and instead of using transformation formulae satisfied by theta functions, the proof requires the knowledge of the functional equation satisfied by the Dirichlet series corresponding to the theta functions.

Before we prove Theorem 5, we recall a few facts about the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$. Following [13, pp. 77–80], define

$$ I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} $$

and

$$ K_\nu(z) = \pi \frac{I_{-\nu}(z) - I_\nu(z)}{2\sin\pi\nu}, $$

where the gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, dt.$$

It can be shown directly from the definition of $I_\nu(z)$ that

$$\frac{d}{dz}\left(z^\nu I_\nu(z)\right) = z^\nu I_{\nu-1}(z) \tag{9}$$

and

$$\frac{d}{dz}\left(z^{-\nu} I_\nu(z)\right) = z^{-\nu} I_{\nu+1}(z). \tag{10}$$

By (9) and (10), it follows that

$$\frac{d}{dz}\left(z^{-\nu} K_\nu(z)\right) = -z^{-\nu} K_{\nu+1}(z). \tag{11}$$

We now introduce a new function

$$\kappa_\nu(x) = (-1)^\nu 2^{\nu+\frac{1}{2}} \frac{K_{\nu+\frac{1}{2}}(2\sqrt{x})}{(2\sqrt{x})^{\nu+\frac{1}{2}}}. \tag{12}$$

Using (11), we immediately deduce that

$$\frac{d\kappa_\nu(x)}{dx} = \kappa_{\nu+1}(x). \tag{13}$$

Now, it can be shown from the definition of $K_\nu(z)$ that [13, p. 80, (13)]

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} \exp(-z).$$

This implies, by Theorem 2, that

$$\int_0^\infty \exp\left(-xs^2 - \frac{1}{s^2}\right) ds = \frac{K_{\frac{1}{2}}(2\sqrt{x})}{(\sqrt{x})^{\frac{1}{2}}} = \kappa_0(x). \tag{14}$$

Differentiating (14) ν times with respect to x, we deduce from (13) that

$$\int_0^\infty (-1)^\nu s^{2\nu} \exp\left(-xs^2 - \frac{1}{s^2}\right) ds = \kappa_\nu(x). \tag{15}$$

In the previous section, we proved Theorem 4 using Theorem 2 and the transformation formula given in Theorem 3. The same method can be used to prove Theorem 5.

Proof of Theorem 5. For any non-negative integer ℓ in (15), we apply the transformation formula to $2\ell + 1$ copies of the theta function associated with $Q(m, n)$, namely,

$$
\begin{aligned}
&\left(\sum_{(m,n)\in\mathbf{Z}^2} \exp\left(\frac{-2\pi t^2 Q(m,n)}{\sqrt{D}}\right)\right)^{2\ell+1} \\
&\qquad = \sum_{(\vec{m},\vec{n})\in\mathbf{Z}^{4\ell+2}} \exp\left(-2\pi t^2 \sum_{j=1}^{2\ell+1} \frac{Q(m_j,n_j)}{\sqrt{D}}\right) \\
&\qquad = \left(\frac{1}{t^2}\right)^{2\ell+1} \sum_{(\vec{m},\vec{n})\in\mathbf{Z}^{4\ell+2}} \exp\left(-2\pi \sum_{j=1}^{2\ell+1} \frac{Q(m_j,n_j)}{t^2\sqrt{D}}\right),
\end{aligned}
\tag{16}
$$

where $\vec{m} = (m_1, \cdots, m_{2\ell+1})$ and $\vec{n} = (n_1, \cdots, n_{2\ell+1})$. We now multiply

$$
(-1)^\ell t^{2\ell} \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\alpha t^2 + \frac{\beta}{t^2}\right)\right)
$$

to both sides of (16) and integrate (with a change of variables on one side) to obtain

$$
\begin{aligned}
&\int_0^\infty (-1)^\ell t^{2\ell} \sum_{(\vec{m},\vec{n})\in\mathbf{Z}^{4\ell+2}} \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\alpha t^2 + \frac{\beta}{t^2}\right) - 2\pi t^2 \sum_{j=1}^{2\ell+1} \frac{Q(m_j,n_j)}{\sqrt{D}}\right) dt \\
&\qquad = \int_0^\infty (-1)^\ell t^{2\ell} \sum_{(\vec{m},\vec{n})\in\mathbf{Z}^{4\ell+2}} \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\frac{\alpha}{t^2} + \beta t^2\right) - 2\pi t^2 \sum_{j=1}^{2\ell+1} \frac{Q(m_j,n_j)}{\sqrt{D}}\right) dt.
\end{aligned}
\tag{17}
$$

Using (15) in (17) with a suitable change of variables on each side, we deduce that

$$
\begin{aligned}
&\sum_{(\vec{m},\vec{n})\in\mathbf{Z}^{4\ell+2}} \left(\frac{2\pi\beta}{\sqrt{D}}\right)^{\frac{2\ell+1}{2}} \kappa_\ell\left(\frac{4\pi^2\beta}{D}\left(\alpha + \sum_{j=1}^{2\ell+1} Q(m_j,n_j)\right)\right) \\
&\qquad = \sum_{(\vec{m},\vec{n})\in\mathbf{Z}^{4\ell+2}} \left(\frac{2\pi\alpha}{\sqrt{D}}\right)^{\frac{2\ell+1}{2}} \kappa_\ell\left(\frac{4\pi^2\alpha}{D}\left(\beta + \sum_{j=1}^{2\ell+1} Q(m_j,n_j)\right)\right).
\end{aligned}
\tag{18}
$$

It remains to connect $\kappa_\ell(z)$ with ${}_2F_0(-\ell, \ell+1; z)$. We note that for non-negative integers ℓ, the expression

$$
z^\ell \, {}_2F_0\left(-\ell, \ell+1; -\frac{1}{2z}\right) \tag{19}
$$

defines a family of polynomials studied by Grosswald [9]. (Grosswald denotes these as $\theta_\ell(z)$ and they are sometimes known as the reverse Bessel polynomials.) A result from [9, p. 34, (1)] connects these polynomials to the modified Bessel functions.

$$
{}_2F_0\left(-\ell, \ell+1; -\frac{1}{2z}\right) = \sqrt{\frac{2z}{\pi}} \exp(z) K_{\ell+\frac{1}{2}}(z). \tag{20}
$$

Using (12) and (20), we deduce that

$$
\kappa_\ell(x) = (-1)^\ell \frac{\sqrt{\pi}}{2} \frac{\exp(-2\sqrt{x})}{(\sqrt{x})^{\ell+1}} \, {}_2F_0\left(-\ell, \ell+1; -\frac{1}{4\sqrt{x}}\right). \tag{21}
$$

Substituting (21) in (18) and simplifying, we complete the proof of Theorem 5. □

Remark 3.1. It is interesting to note that although Grosswald wrote a book on representing integers as sums of squares [10] and another book on Bessel polynomials [9], neither Theorem 5 nor its specialization to $r_{4\ell+2}(n)$ were mentioned in either of his books. Theorem 5 definitely serves as a good bridge between the two books of Grosswald.

We would like to point out that other analogues of Theorem 5 can be obtained with $r_{Q,2\ell+1}(n)$ replaced by some arithmetic function that satisfy a transformation formula analogous to Theorem 3. For example, the Dedekind eta function is defined as

$$\eta(\tau) = \exp\left(\frac{\pi i\tau}{12}\right)\prod_{n=1}^{\infty}\left(1-\exp(2\pi i n\tau)\right),$$

for some τ in the complex upper half plane. If we take $\tau = it$ for some positive real number t, it is known that

$$\eta^2(it) = \frac{1}{t}\eta^2\left(\frac{i}{t}\right). \tag{22}$$

So if we define the following function using $q = \exp(2\pi i\tau)$

$$\sum_{n=0}^{\infty} f_{4\ell+2}(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{4\ell+2}, \tag{23}$$

we can then obtain in a similar fashion the following analogue of Theorem 5.

Theorem 6. *Let ℓ be a non-negative integer. Then*

$$\sum_{n\in\mathbf{Z}} f_{4\ell+2}(n)\frac{\exp\left(-4\pi\sqrt{\left(n+\frac{2\ell+1}{12}+\alpha\right)\beta}\right)\sqrt{\beta^{\ell}}}{\sqrt{\left(n+\frac{2\ell+1}{12}+\alpha\right)^{\ell+1}}}\,{}_2F_0\left(-\ell,\ell+1;\frac{-1}{8\pi\sqrt{(n+\frac{2\ell+1}{12}+\alpha)\beta}}\right)$$
$$= \sum_{n\in\mathbf{Z}} f_{4\ell+2}(n)\frac{\exp\left(-4\pi\sqrt{\left(n+\frac{2\ell+1}{12}+\beta\right)\alpha}\right)\sqrt{\alpha^{\ell}}}{\sqrt{\left(n+\frac{2\ell+1}{12}+\beta\right)^{\ell+1}}}\,{}_2F_0\left(-\ell,\ell+1;\frac{-1}{8\pi\sqrt{(n+\frac{2\ell+1}{12}+\beta)\alpha}}\right).$$

Since we have already given examples of Theorem 5 for $\ell = 0$ and $\ell = 1$, we shall end by giving an example of Theorem 6 in the case $\ell = 2$. Now

$${}_2F_0(-2,3;z) = 12z^2 - 6z + 1.$$

From [6, Th 4.2], a two variable series representation for $\eta^{10}(\tau)$ is given by

$$\eta^{10}(\tau) = q^{5/12}\sum_{n=0}^{\infty} f_{10}(n)q^n = \frac{1}{6}\sum_{\substack{m\equiv 1 \pmod 6\\ n\equiv 4 \pmod 6}} mn(m^2-n^2)q^{\frac{m^2+n^2}{12}}.$$

Other two variable representations for $\eta^2(\tau)$, $\eta^6(\tau)$, $\eta^{14}(\tau)$ and $\eta^{26}(\tau)$ can also be found in [6,5]. For $\eta^{10}(\tau)$, the following holds.

Corollary 3.2. *For any positive real numbers* α *and* β*,*

$$\sum_{\substack{m\equiv 1 \pmod 6\\ n\equiv 4 \pmod 6}} mn(m^2-n^2)\frac{\exp\left(-4\pi\sqrt{\left(\frac{m^2+n^2}{12}+\alpha\right)\beta}\right)}{\sqrt{\left(\frac{m^2+n^2}{12}+\alpha\right)^5}} \times\left(3+12\pi\sqrt{\left(\frac{m^2+n^2}{12}+\alpha\right)\beta}+16\pi^2\left(\frac{m^2+n^2}{12}+\alpha\right)\beta\right)$$
$$=\sum_{\substack{m\equiv 1 \pmod 6\\ n\equiv 4 \pmod 6}} mn(m^2-n^2)\frac{\exp\left(-4\pi\sqrt{\left(\frac{m^2+n^2}{12}+\beta\right)\alpha}\right)}{\sqrt{\left(\frac{m^2+n^2}{12}+\beta\right)^5}} \times\left(3+12\pi\sqrt{\left(\frac{m^2+n^2}{12}+\beta\right)\alpha}+16\pi^2\left(\frac{m^2+n^2}{12}+\beta\right)\alpha\right).$$

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