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FINE'S FUNCTION AND PARTIAL THETA FUNCTION

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(Communicated by Mourad Ismail)

Dedicated to the memory of Professor Richard Askey

ABSTRACT. In this article, we use identities found in N.J. Fine's book *Basic hypergeometric series and its applications* to derive a one-parameter generalization of the product of two partial theta functions discovered by G.E. Andrews and S.O. Warnaar. We also give two different proofs of this generalization, one of which is motivated by the work of A. Berkovich and the other is given by M.E.H. Ismail.

1. INTRODUCTION

Around 2003, S.O. Warnaar discovered a very beautiful formula [9, Theorem 1.5] given by

$$1+\sum_{j=1}^{\infty}(-1)^j q^{j(j-1)/2}(a^j+b^j)=(q)_\infty(a)_\infty(b)_\infty\sum_{j=0}^{\infty}\frac{(ab/q)_{2j}q^j}{(q)_j(a)_j(b)_j(ab)_j}, \tag{1.1}$$

where $(a)_0=(a;q)_0=1$ and for $n\geq 1$,

$$(a)_n=(a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1}),$$

and

$$(a)_\infty=(a;q)_\infty=\prod_{k=1}^{\infty}(1-aq^{k-1}).$$

When $b=q/a$, the sum on the right hand side of (1.1) is 1 since for $n\geq 1$,

$$(ab/q)_n=(1)_n=(1-1)(1-q)\cdots(1-q^{n-1})=0.$$

The identity (1.1) then reduces to the famous Jacobi triple product identity which states that for $a,q\in\mathbf{C}$ with $a\neq 0$ and $|q|<1$,

$$\sum_{j=-\infty}^{\infty}(-1)^j a^j q^{j(j-1)/2}=(q;q)_\infty(a;q)_\infty(q/a;q)_\infty. \tag{1.2}$$

Warnaar's proof of (1.1) involves the simplification of a five-fold sum using q-Chu-Vandermonde identity and appears to be the only available proof until 2007, when

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G.E. Andrews and Warnaar discovered another very beautiful formula for which (1.1) follows as a corollary. The formula of Andrews and Warnaar is given by

(1.3)
$$\left(\sum_{j=0}^{\infty}(-1)^j a^j q^{j(j-1)/2}\right)\left(\sum_{j=0}^{\infty}(-1)^j b^j q^{j(j-1)/2}\right) = (q)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty}\frac{(abq^{j-1})_j q^j}{(q)_j (a)_j (b)_j}.$$

Andrews and Warnaar concluded their article showing that (1.3) follows from (1.1). For a more detailed survey of (1.3) and (1.1), we refer the readers to the book by Andrews and B.C. Berndt [1, p. 142].

In this short note, we will derive an extension of (1.3). The identity is given as follows:

Theorem 1. *Let $a, b, q, t \in \mathbf{C}$ with $|q| < 1, |t| < 1, a \neq 0$ and $b \neq 0$. Then*

$$\left(\sum_{j=0}^{\infty}\frac{(t)_j}{(q)_j}(-1)^j a^j q^{j(j-1)/2}\right)\left(\sum_{j=0}^{\infty}\frac{(t)_j}{(q)_j}(-1)^j b^j q^{j(j-1)/2}\right) = (t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty}\frac{(abq^{j-1})_j t^j}{(q)_j (a)_j (b)_j}. \tag{1.4}$$

Note that the generalization (1.4) is surprisingly easy to remember if one has (1.3). One simply inserts $(t)_j/(q)_j$ on the two sums on the left hand side and replace $(q)_\infty$ and q^j by $(t)_\infty$ and t^j on the right hand side, respectively. In the next section, we will introduce Fine's function and give a proof of Theorem 1.

In Section 3, we express (1.4) in terms of ${}_2\phi_1$ and ${}_4\phi_3$ and obtain (3.4). Identity (3.4) is a generalization of an identity found by A. Berkovich [3, (2.5)]. We prove (3.4) which in turn gives a second proof of Theorem 1. In Section 4, we present a third proof of Theorem 1 due to M.E.H. Ismail [8]. We conclude the article with Ismail's interesting observation which identifies (3.4) as a q-analogue of an identity involving a product of two Bessel functions.

2. Fine's function and Theorem 1

The first chapter of N.J. Fine's important book [5] is devoted to the study of the function

$$F(a, b; t) = 1 + \sum_{j=1}^{\infty}\frac{(aq)_j}{(bq)_j}t^j.$$

For the ease of reference to this function, we will call $F(a, b; t)$ Fine's function. Many well-known identities such as the Rogers-Fine identity [5, Section 4], the Jacobi triple product identity [5, Section 17] and the Ramanujan ${}_1\psi_1$ summation formula [5, Section 18] were featured and proved using Fine's function. There are many more lesser known, yet interesting identities in Fine's book. A few of these identities are related to (1.1). For example, one finds, with slight modifications, that [5, (7.32)]

$$\sum_{j=0}^{\infty}\frac{q^j}{(c)_j (q)_j} = \frac{1}{(c)_\infty (q)_\infty}\sum_{j=0}^{\infty}(-1)^j c^j q^{(j^2-j)/2} \tag{2.1}$$

and [5, (20.52)]

$$\sum_{j=0}^{\infty}\frac{(t)_j}{(q)_j}(-1)^jc^jq^{(j^2-j)/2}=(t)_\infty(c)_\infty\sum_{j=0}^{\infty}\frac{t^j}{(q)_j(c)_j} \tag{2.2}$$

$$=(c)_\infty\sum_{j=0}^{\infty}\frac{(ct)^jq^{j^2-j}}{(q)_j(c)_j}. \tag{2.3}$$

Observe that the right hand side of (2.1) appears in (1.1) and (1.3). It is this observation that motivated us to search for the connection between Fine's function $F(a,b;q)$ and the Andrews-Warnaar identity. We were led to Theorem 1 using (2.2) and (2.3). We now present the proof of Theorem 1, which is surprisingly identical to the proof of (1.3) presented in [2].

Proof of Theorem 1. Applying (2.2) with $c=a$ and (2.3) with $c=b$, we deduce that

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty}\frac{(t)_j}{(q)_j}(-1)^ja^jq^{j(j-1)/2}\right)\left(\sum_{\ell=0}^{\infty}\frac{(t)_\ell}{(q)_\ell}(-1)^\ell b^\ell q^{\ell(\ell-1)/2}\right)\\
&=(t)_\infty(a)_\infty(b)_\infty\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{t^{j+\ell}q^{\ell^2-\ell}b^\ell}{(q)_j(a)_j(q)_\ell(b)_\ell}\\
&=(t)_\infty(a)_\infty(b)_\infty\sum_{m=0}^{\infty}\frac{t^mb^mq^{m^2-m}}{(q)_m(b)_m}\sum_{j=0}^{m}\frac{q^{j^2-2mj+j}b^{-j}(q^{m-j+1})_j(bq^{m-j})_j}{(q)_j(a)_j}\\
&=(t)_\infty(a)_\infty(b)_\infty\sum_{m=0}^{\infty}\frac{t^mb^mq^{m^2-m}}{(q)_m(b)_m}\sum_{j=0}^{m}\frac{q^j(q^{-m})_j(q^{1-m}/b)_j}{(q)_j(a)_j}\\
&=(t)_\infty(a)_\infty(b)_\infty\sum_{m=0}^{\infty}\frac{t^m(abq^{m-1})_m}{(q)_m(a)_m(b)_m},
\end{aligned}$$

where we have used the q-Chu-Vandemonde identity [6, (1.5.3)] to show that

$$\sum_{j=0}^{m}\frac{q^j(q^{-m})_j(q^{1-m}/b)_j}{(q)_j(a)_j}=\frac{(abq^{m-1})_m}{(a)_m}b^{-m}q^{m-m^2}. \qquad \square$$

In [2], Andrews and Warnaar derived (1.1) from (1.3). It is natural to try and derive an "analogue" of (1.1) from (1.4). In view of (1.1), it is therefore appropriate to seek for an expression for

$$1+\sum_{j=1}^{\infty}\frac{(t)_j}{(q)_j}(-1)^jq^{j(j-1)/2}\big(a^j+b^j\big).$$

We have not been able to find an alternate expression for the above. However, we are able to derive an identity involving the one-variable extension of

$$(q)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{q^j (ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j},$$

which is the right-hand side of (1.1). The identity is stated as follows.

Theorem 2. *Let* $a, b, q, t \in \mathbf{C}$ *with* $|q| < 1$. *Then*

$$\begin{aligned} &(t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{t^j (ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j} \\ &= \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j a^j q^{j(j-1)/2} \right) \left(\sum_{\ell=0}^{\infty} \frac{(t)_\ell}{(q)_\ell} (-1)^\ell b^\ell q^{\ell(\ell-1)/2} \right) \\ &\quad - \frac{t}{q} \left(\sum_{j=1}^{\infty} \frac{(t)_{j-1}}{(q)_{j-1}} (-1)^j a^j q^{j(j-1)/2} \right) \left(\sum_{\ell=1}^{\infty} \frac{(t)_{\ell-1}}{(q)_{\ell-1}} (-1)^\ell b^\ell q^{\ell(\ell-1)/2} \right). \end{aligned} \tag{2.4}$$

Proof. We follow the steps in the proof of (1.3) given in [2] and write

$$\begin{aligned} &(t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j} t^j \\ &= (t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(1 - ab/q)(abq^j)_{j-1}}{(q)_j (a)_j (b)_j} t^j \\ &= (t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(1 - abq^{j-1} - ab/q(1 - q^j))(abq^j)_{j-1}}{(q)_j (a)_j (b)_j} t^j \\ &= (t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(abq^{j-1})_j}{(q)_j (a)_j (b)_j} t^j \\ &\qquad - (t)_\infty (a)_\infty (b)_\infty ab \sum_{j=1}^{\infty} \frac{(abq^j)_{j-1}}{(q)_{j-1} (a)_j (b)_j} \frac{t^j}{q} \\ &= (t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(abq^{j-1})_j}{(q)_j (a)_j (b)_j} t^j \\ &\qquad - (t)_\infty (aq)_\infty (bq)_\infty \frac{abt}{q} \sum_{\ell=0}^{\infty} \frac{(abq^{\ell+1})_\ell}{(q)_\ell (aq)_\ell (bq)_\ell} t^\ell. \end{aligned} \tag{2.5}$$

By (1.4) and (2.5), we deduce that

$$
\begin{aligned}
&(t)_\infty(a)_\infty(b)_\infty \sum_{j=0}^{\infty} \frac{t^j (ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j} \\
&= \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j a^j q^{j(j-1)/2}\right) \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j b^j q^{j(j-1)/2}\right) \\
&\quad - \frac{abt}{q} \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j (aq)^j q^{j(j-1)/2}\right) \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j (bq)^j q^{j(j-1)/2}\right) \\
&= \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j a^j q^{j(j-1)/2}\right) \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j b^j q^{j(j-1)/2}\right) \\
&\quad - \frac{t}{q} \left(\sum_{j=1}^{\infty} \frac{(t)_{j-1}}{(q)_{j-1}} (-1)^j a^j q^{j(j-1)/2}\right) \left(\sum_{j=1}^{\infty} \frac{(t)_{j-1}}{(q)_{j-1}} (-1)^j b^j q^{j(j-1)/2}\right). \quad \square
\end{aligned}
\tag{2.6}
$$

When $t = q$, the term $(t)_{n-1}/(q)_{n-1}$ is 1 and (2.4) reduces to

$$
\begin{aligned}
&(q)_\infty(a)_\infty(b)_\infty \sum_{j=0}^{\infty} \frac{(ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j} q^j \\
&= \left(\sum_{j=0}^{\infty} (-1)^j a^j q^{j(j-1)/2}\right) \left(\sum_{\ell=0}^{\infty} (-1)^\ell b^\ell q^{\ell(\ell-1)/2}\right) \\
&\quad - \left(\sum_{j=1}^{\infty} (-1)^j a^j q^{j(j-1)/2}\right) \left(\sum_{\ell=1}^{\infty} (-1)^\ell b^\ell q^{\ell(\ell-1)/2}\right).
\end{aligned}
$$

Writing

$$
A = \sum_{j=1}^{\infty} (-1)^j a^j q^{j(j-1)/2} \text{ and } B = \sum_{\ell=1}^{\infty} (-1)^\ell b^\ell q^{\ell(\ell-1)/2}
$$

yields

$$
\begin{aligned}
&(q)_\infty(a)_\infty(b)_\infty \sum_{j=0}^{\infty} \frac{(ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j} q^j \\
&= (1+A)(1+B) - AB = 1 + A + B,
\end{aligned}
$$

which is Warnaar's identity (1.1). This is how Andrews and Warnaar derived (1.1) from (1.3).

Let

$$
M(a, b, t : q) = \left(\sum_{j=0}^{\infty} \frac{(t)_j}{(q)_j} (-1)^j a^j q^{j(j-1)/2}\right) \left(\sum_{\ell=0}^{\infty} \frac{(t)_\ell}{(q)_\ell} (-1)^\ell b^\ell q^{\ell(\ell-1)/2}\right).
$$

Then by (2.6), we rewrite (2.4) as

$$(t)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j} t^j = M(a,b,t:q) - \frac{abt}{q} M(aq,bq,t:q).$$

This is the generalization of

$$(q)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab)_j} q^j = M(a,b,q:q) - abM(aq,bq,q:q),$$

which appears in the proof of (1.1) in [2].

3. Basic hypergeometric series and (1.3)

It was known from [2] that

$$(q)_\infty (a)_\infty (b)_\infty \sum_{j=0}^{\infty} \frac{(abq^{j-1})_j}{(q)_j (a)_j (b)_j} q^j = \sum_{j=0}^{\infty} \frac{(ab/q)_{2j}}{(q)_j (a)_j (b)_j (ab/q)_j} q^j$$
$$= (q)_\infty (a)_\infty (b)_\infty \, {}_4\phi_3 \left(\begin{matrix} \sqrt{ab/q}, -\sqrt{ab/q}, \sqrt{ab}, -\sqrt{ab} \\ a, b, ab/q \end{matrix} ; q, q \right),$$

where for $r \in \mathbf{Z}^+$,

$${}_r\phi_{r-1} \left(\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_{r-1} \end{matrix} ; q, z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j \cdots (a_r)_j}{(b_1)_j (b_2)_j \cdots (b_{r-1})_j} \frac{z^j}{(q)_j}.$$

This motivated Berkovich [3] to show that

$$\sum_{j=0}^{\infty} (-1)^j q^{j(j-1)/2} a^j = \lim_{\rho \to \infty} {}_2\phi_1 \left(\begin{matrix} \rho, q \\ 1/\rho \end{matrix} ; q, a/\rho \right) = (q)_\infty (a)_\infty \, {}_2\phi_1 \left(\begin{matrix} 0, 0 \\ a \end{matrix} ; q, q \right), \tag{3.1}$$

where the last equality follows from Heine's transformation formula

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \, {}_2\phi_1 \left(\begin{matrix} c/b, z \\ az \end{matrix} ; q, b \right).$$

He then concluded that (1.3) is equivalent to

$$\begin{aligned} &{}_2\phi_1 \left(\begin{matrix} 0, 0 \\ a \end{matrix} ; q, q \right) {}_2\phi_1 \left(\begin{matrix} 0, 0 \\ b \end{matrix} ; q, q \right) \\ &\quad = \frac{1}{(q)_\infty} \, {}_4\phi_3 \left(\begin{matrix} \sqrt{ab/q}, -\sqrt{ab/q}, \sqrt{ab}, -\sqrt{ab} \\ a, b, ab/q \end{matrix} ; q, q \right) \end{aligned} \tag{3.2}$$

and deduced (3.2) from the Gasper-Rahman product formula [6, (8.8.18)]

$$\begin{aligned} &{}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) {}_2\phi_1 \left(\begin{matrix} a, aq/c \\ aq/b \end{matrix} ; q, z \right) \\ &\quad = \frac{(az)_\infty (abz/c)_\infty}{(z)_\infty (bz/c)_\infty} \, {}_6\phi_5 \left(\begin{matrix} a, c/b, \sqrt{ac/b}, -\sqrt{ac/b}, \sqrt{acq/b}, -\sqrt{acq/b} \\ aq/b, c, ac/b, az, cq/(bz) \end{matrix} ; q, q \right) \\ &\qquad + \frac{(a)_\infty (c/b)_\infty (az)_\infty (bz)_\infty (azq/c)_\infty}{(c)_\infty (aq/b)_\infty (z)_\infty (z)_\infty (c/bz)_\infty} \\ &\qquad \times \, {}_6\phi_5 \left(\begin{matrix} z, abz/c, z\sqrt{ab/c}, -z\sqrt{ab/c}, z\sqrt{abq/c}, -z\sqrt{abq/c} \\ az, bz, azq/c, bzq/c, abz^2/c \end{matrix} ; q, q \right), \end{aligned} \tag{3.3}$$

where $|z| < 1$.

Using exactly the same argument as in (3.1), we deduce that

$$\begin{aligned}\sum_{j=0}^{\infty}(-1)^j \frac{(t)_j}{(q)_j} q^{j(j-1)/2} a^j &= \lim_{\rho\to\infty} {}_2\phi_1\left(\begin{matrix}\rho, t\\ 1/\rho\end{matrix}; q, a/\rho\right)\\ &= \lim_{\rho\to\infty} \frac{(t)_\infty (a)_\infty}{(1/\rho)_\infty (a/\rho)_\infty}\, {}_2\phi_1\left(\begin{matrix}1/\rho t, a/\rho\\ a\end{matrix}; q, t\right)\\ &= (t)_\infty (a)_\infty\, {}_2\phi_1\left(\begin{matrix}0,0\\ a\end{matrix} q, t\right).\end{aligned}$$

This shows that (1.4) is equivalent to the following identity.

Theorem 3. *For* $|t| < 1$,

$$\begin{aligned}&{}_2\phi_1\left(\begin{matrix}0,0\\ a\end{matrix}; q, t\right) {}_2\phi_1\left(\begin{matrix}0,0\\ b\end{matrix}; q, t\right)\\ &= \frac{1}{(t)_\infty}\, {}_4\phi_3\left(\begin{matrix}\sqrt{ab/q}, -\sqrt{ab/q}, \sqrt{ab}, -\sqrt{ab}\\ a, b, ab/q\end{matrix}; q, t\right).\end{aligned} \tag{3.4}$$

Identity (3.4) is an example of a q-analogue of Clausen's identities (see [6, (8.8.1)]) for hypergeometric series. We now give a proof of (3.4) using (3.3), which yields the second proof of (1.4).

Proof. Let $a = bd/q$ and let $b \to 0$. Then the left-hand side of (3.3) yields

$$ {}_2\phi_1\left(\begin{matrix}0,0\\ c\end{matrix}; q, t\right) {}_2\phi_1\left(\begin{matrix}0,0\\ d\end{matrix}; q, t\right). \tag{3.5}$$

The first term of the right-hand side of (3.3) gives

$$\frac{1}{(z)_\infty} \lim_{b\to 0} {}_6\phi_5\left(\begin{matrix}bd/q, c/b, \sqrt{cd/q}, -\sqrt{cd/q}, \sqrt{cd}, -\sqrt{cd}\\ d, c, cd/q, bdz/q, cq/(bz)\end{matrix}; q, q\right). \tag{3.6}$$

Since

$$\lim_{b\to 0} \frac{(c/b)_n}{(cq/(bz))_n} q^n = \lim_{b\to 0} \frac{(b-c)(b-cq)\cdots(b-cq^{n-1})z^n}{(bz-cq)(bz-cq^2)\cdots(bz-cq^{n-1})(bz-cq^n)} q^n = z^n,$$

we deduce from (3.6) that the first term of the right-hand side of (3.3) is

$$\frac{1}{(z)_\infty}\, {}_4\phi_3\left(\begin{matrix}\sqrt{cd/q}, -\sqrt{cd/q}, \sqrt{cd}, -\sqrt{cd}\\ c, d, cd/q\end{matrix}; q, z\right). \tag{3.7}$$

For the second term of the right-hand side of (3.3), we observe that

$$\lim_{b\to 0} \frac{(c/b)_\infty}{(c/(bz))_\infty} = \lim_{n\to\infty} \lim_{b\to 0} \frac{(b-c)(b-cq)\cdots(b-cq^{n-1})}{(bz-c)(bz-cq)\cdots(bz-cq^{n-1})} z^n = 0,$$

since $|z| < 1$.

Combining (3.7) and the vanishing of the second term of the right-hand side of (3.3), we obtain (3.4) after replacing (c, d, z) by (a, b, t). □

It is easy to remember (3.4) if we are given (3.2). The left-hand side of (3.4) is simply the left-hand side of (3.2) with q in the last argument of the two ${}_2\phi_1$ replaced by t and the right-hand side of (3.4) is the right-hand side of (3.2) with $(q)_\infty$ by $(t)_\infty$ and q in the last argument of ${}_4\phi_3$ by t.

4. Third proof of (1.4)

In this section, we give a third proof of (1.4). This proof is due to Ismail [8] and it is independent of (2.2), (2.3) and (3.3). We first divide (1.4) by $(t)_\infty$ on both sides to arrive at

$$\begin{aligned}\left(\sum_{j=0}^{\infty}\frac{(t)_j}{(t)_\infty(q)_j}(-1)^j a^j q^{j(j-1)/2}\right)\left(\sum_{j=0}^{\infty}\frac{(t)_j}{(q)_j}(-1)^j b^j q^{j(j-1)/2}\right)\\ = (a)_\infty(b)_\infty\sum_{j=0}^{\infty}\frac{(abq^{j-1})_j t^j}{(q)_j(a)_j(b)_j}.\end{aligned} \tag{4.1}$$

The left hand side of (4.1) is

$$\begin{aligned}&\left(\sum_{j=0}^{\infty}\frac{(t)_j}{(t)_\infty(q)_j}(-1)^j a^j q^{j(j-1)/2}\right)\left(\sum_{k=0}^{\infty}\frac{(t)_k}{(q)_k}(-1)^k b^k q^{k(k-1)/2}\right)\\ &= \left(\sum_{j=0}^{\infty}\frac{1}{(q)_j}(-1)^j a^j q^{j(j-1)/2}\frac{1}{(tq^j)_\infty}\right)\left(\sum_{k=0}^{\infty}\frac{1}{(q)_k}(-1)^k b^k q^{k(k-1)/2}\frac{(t)_\infty}{(tq^k)_\infty}\right).\end{aligned} \tag{4.2}$$

By Euler's theorem (see [6, Eq. (II.1), p. 354]),

$$\frac{1}{(tq^j)_\infty} = \sum_{r=0}^{\infty}\frac{t^r q^{jr}}{(q)_r} \tag{4.3}$$

and by the q-binomial theorem [6, Eq. (II.3), p. 354]

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}z^n = \frac{(az)_\infty}{(z;q)_\infty}$$

with $a = q^{-k}$ and $z = tq^k$, we have

$$\begin{aligned}\frac{(t)_\infty}{(tq^k)_\infty} &= \sum_{s=0}^{\infty}\frac{(q^{-k})_s}{(q)_s}t^s q^{ks}\\ &= \sum_{s=0}^{k}\frac{(q^{k+1-s})_s}{(q)_s}(-1)^s t^s q^{s(s-1)/2}\end{aligned} \tag{4.4}$$

since

$$(q^{-k})_s = (1-q^{-k})(1-q^{1-k})\cdots(1-q^{s-1-k}) = (-1)^k q^{s(s-1)/2-ks}(q^{k+1-s})_s.$$

Substituting (4.3) and (4.4) into (4.2), we see that the left hand side of (4.1) is equivalent to

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty}\frac{1}{(q)_j}(-1)^j a^j q^{j(j-1)/2}\sum_{r=0}^{\infty}\frac{t^r q^{jr}}{(q)_r}\right)\\
&\times\left(\sum_{k=0}^{\infty}\frac{1}{(q)_k}(-1)^k b^k q^{k(k-1)/2}\sum_{s=0}^{k}\frac{(q^{k+1-s})_s}{(q)_s}(-1)^s t^s q^{s(s-1)/2}\right)\\
&=\left(\sum_{r=0}^{\infty}\frac{t^r}{(q)_r}\sum_{j=0}^{\infty}\frac{(-1)^j a^j q^{jr+j(j-1)/2}}{(q)_j}\right)\\
&\quad\times\left(\sum_{s=0}^{\infty}\frac{(-1)^s t^s q^{s(s-1)/2}}{(q)_s}\sum_{k=s}^{\infty}\frac{1}{(q)_{k-s}}(-1)^k b^k q^{k(k-1)/2}\right)\\
&=\left(\sum_{r=0}^{\infty}\frac{t^r}{(q)_r}\sum_{j=0}^{\infty}\frac{(-1)^j a^j q^{jr+j(j-1)/2}}{(q)_j}\right)\\
&\quad\times\left(\sum_{s=0}^{\infty}\frac{b^s t^s q^{s(s-1)}}{(q)_s}\sum_{k=0}^{\infty}\frac{(-1)^k b^k q^{ks+k(k-1)/2}}{(q)_k}\right)\\
&=\left(\sum_{r=0}^{\infty}\frac{t^r}{(q)_r}(aq^r)_\infty\right)\left(\sum_{s=0}^{\infty}\frac{b^s t^s q^{s(s-1)}}{(q)_s}(bq^s)_\infty\right)\\
&=(a)_\infty(b)_\infty\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{b^s q^{s(s-1)}}{(a)_r(q)_r(b)_s(q)_s}t^{r+s}
\end{aligned}$$

$$=(a)_\infty(b)_\infty\sum_{m=0}^{\infty}t^m\sum_{s=0}^{m}\frac{b^s q^{s(s-1)}}{(a)_{m-s}(q)_{m-s}(b)_s(q)_s}, \tag{4.5}$$

where in the third equation, we applied Euler's Theorem [6, Eq. (II.2), p. 354]

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q)_n}z^n=(-z)_\infty$$

once in each of the parentheses, and in the final equality, we sum over $m=r+s$.

The inner sum in (4.5) is

$$\begin{aligned}
\sum_{s=0}^{m}\frac{b^s q^{s(s-1)}}{(a)_{m-s}(q)_{m-s}(b)_s(q)_s}&=\sum_{s=0}^{m}\frac{(aq^{m-s})_s(q^{m-s+1})_s b^s q^{s(s-1)}}{(a)_m(q)_m(b)_s(q)_s}\\
&=\frac{1}{(a)_m(q)_m}\sum_{s=0}^{m}\frac{(q^{1-m}/a)_s(q^{-m})_s a^s b^s q^{(2m-1)s}}{(b)_s(q)_s}
\end{aligned}$$

$$=\frac{(abq^{m-1})_m}{(a)_m(b)_m(q)_m}, \tag{4.6}$$

where in the last equality, we used the q-Chu-Vandermonde sum [6, Eq. (II.7), p. 354]

$$\sum_{n=0}^{m}\frac{(d)_n(q^{-m})_n}{(c)_n(q)_n}(cq^m/d)^n=\frac{(c/d)_m}{(c)_m}$$

with $c=b$ and $d=q^{1-m}/a$.

Substituting (4.6) into (4.5), we obtain the right hand side of (4.1), and this completes Ismail's proof of (1.4).

5. Ismail's important observation

In this final section, we share an important observation by Ismail [8]. First, we recall that the shifted factorial is

$$(a)_0 = 1,$$
$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n = 1, 2, \ldots.$$

Note that these share the same notation as the *q-shifted factorial* given in Section 1. To avoid ambiguity, for this section, $(a)_n$ is used exclusively for the shifted factorial, while we add in the base q and use the notation $(a;q)_n$ when referring to the q-products.

The Bessel function of the first kind [4, Eqs. (2), (3), p. 4] is given by

$$\begin{aligned} J_\nu(z) &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^{2n+\nu}\frac{1}{m!\Gamma(m+\nu+1)} \\ &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left({- \atop \nu+1}; -\frac{z^2}{4}\right), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} {}_rF_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; z) &= {}_rF_s\left({a_1, a_2, \ldots, a_r \atop b_1, \ldots, b_s}; z\right) \\ &= \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}z^n, \end{aligned}$$

and the — in (5.1) indicates the absence of the numerator.

The q-analog of $J_\nu(z)$ introduced by Jackson, as given in Ismail's article [7, Eq. (14.1.16)], is

$$J_\nu^{(1)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{\nu+2n}}{(q^{\nu+1};q)_n(q;q)_n}.$$

Ismail observed that ${}_2\phi_1(0,0;a;q,t)$ is essentially a q-Bessel function $J_\nu^{(1)}(z;q)$. Indeed, setting $a = q^{\nu+1}$ and $t = -z^2/4$ gives

$${}_2\phi_1\left({0, 0 \atop q^{\nu+1}}; q, -\frac{z^2}{4}\right) = \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty}\left(\frac{2}{z}\right)^\nu J_\nu^{(1)}(z;q).$$

Thus Theorem 3 is a theorem on a product of two q-Bessel functions.

Ismail also showed us that (3.4) is precisely the q-analog of the following identity involving Bessel functions [4, Eq. (49), p. 11].

Corollary 1. *We have*

$$\Gamma(\nu+1)\Gamma(\mu+1)J_\nu(z)J_\mu(z) = (z/2)^{\nu+\mu}\,{}_2F_3\left({(\nu+\mu+1)/2, (\nu+\mu)/2+1 \atop \nu+1, \mu+1, \nu+\mu+1}; -z^2\right). \tag{5.2}$$

Proof. Replacing t by $t(1-q)^2$ and setting $(a,b)=(q^{\nu+1},q^{\mu+1})$, we find that (3.4) becomes

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix}0,0\\ q^{\nu+1}\end{matrix};q,t(1-q)^2\right){}_2\phi_1\left(\begin{matrix}0,0\\ q^{\mu+1}\end{matrix};q,t(1-q)^2\right)\\
&=\frac{1}{(t(1-q)^2;q)_\infty}\\
&\quad\times{}_4\phi_3\left(\begin{matrix}q^{(\nu+\mu+1)/2},-q^{(\nu+\mu+1)/2},q^{(\nu+\mu)/2+1},-q^{(\nu+\mu)/2+1}\\ q^{\nu+1},q^{\mu+1},q^{\nu+\mu+1}\end{matrix};q,t(1-q)^2\right).
\end{aligned}
\tag{5.3}
$$

Note that in taking the limit $q\to 1^-$,

$$
\lim_{q\to 1^-}\frac{(q^a;q)_n}{(1-q)^n}=(a)_n \qquad\text{and}\qquad \lim_{q\to 1^-}(-q^a;q)_n=2^n.
$$

Therefore, letting $q\to 1^-$ in equation (5.3) and setting $t=-z^2/4$, we obtain

$$
{}_0F_1\left(\begin{matrix}-\\ \nu+1\end{matrix};-\frac{z^2}{4}\right){}_0F_1\left(\begin{matrix}-\\ \mu+1\end{matrix};-\frac{z^2}{4}\right)={}_2F_3\left(\begin{matrix}(\nu+\mu+1)/2,(\nu+\mu)/2+1\\ \nu+1,\mu+1,\nu+\mu+1\end{matrix};-z^2\right).
\tag{5.4}
$$

Applying (5.1) twice on the left hand side of (5.4), and rearranging, we obtain (5.2). This completes the proof of the corollary. ☐

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References

[1] George E. Andrews and Bruce C. Berndt, *Ramanujan's lost notebook. Part II*, Springer, New York, 2009. MR2474043

[2] George E. Andrews and S. Ole Warnaar, *The product of partial theta functions*, Adv. in Appl. Math. **39** (2007), no. 1, 116–120, DOI 10.1016/j.aam.2005.12.003. MR2319567

[3] Alexander Berkovich, *On the Difference of Partial Theta Functions*, Bull. Malays. Math. Sci. Soc. **44** (2021), no. 2, 563–570, DOI 10.1007/s40840-020-00961-4. MR4217074

[4] H. van Haeringen and L. P. Kok, *Table errata: Higher transcendental functions, Vol. I [McGraw-Hill, New York, 1953; MR* **15***, 419] by A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi*, Math. Comp. **41** (1983), no. 164, 778. MR717721

[5] Nathan J. Fine, *Basic hypergeometric series and applications*, Mathematical Surveys and Monographs, vol. 27, American Mathematical Society, Providence, RI, 1988. With a foreword by George E. Andrews. MR956465

[6] George Gasper and Mizan Rahman, *Basic hypergeometric series*, 2nd ed., Encyclopedia of Mathematics and Its Applications, vol. 96, Cambridge University Press, Cambridge, 2004. With a foreword by Richard Askey. MR2128719

[7] Mourad E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of Mathematics and Its Applications, vol. 98, Cambridge University Press, Cambridge, 2009. With two chapters by Walter Van Assche; With a foreword by Richard A. Askey; Reprint of the 2005 original. MR2542683

[8] Mourad E. H. Ismail, *private communication.*

[9] S. Ole Warnaar, *Partial theta functions. I. Beyond the lost notebook*, Proc. London Math. Soc. (3) **87** (2003), no. 2, 363–395, DOI 10.1112/S002461150201403X. MR1990932

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