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Borweins' cubic theta functions revisited

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Abstract
Around 1991, J. M. Borwein and P. B. Borwein introduced three cubic theta functions $a(q)$, $b(q)$ and $c(q)$ and discovered many interesting identities associated with these functions. The cubic theta functions $b(q)$ and $c(q)$ have product representations and these representations were first established using the theory of modular forms. The first elementary proof of the product representation of $b(q)$ was discovered in 1994 by the Borweins and F. G. Garvan using one of Euler's identity. They then derived the product representation of $c(q)$ using transformation formulas of Dedekind's $\eta(\tau)$ and some elementary identities satisfied by $a(q)$, $b(q)$ and $c(q)$. In this note, we present three proofs of the product representation of $c(q)$ without the use of the transformation of Dedekind's η-function. We also discuss the connections between these proofs and the works of Baruah and Nath (Proc Am Math Soc 142:441–448, 2014) and Ye (Int J Number Theory 12(7):1791–1800, 2016). We also adopt the idea of the Borweins and Garvan to derive the product representation of Jacobi theta function $\vartheta_4(0|\tau)$ which leads to a proof of the Jacobi triple product identity.

Keywords Cubic theta functions · Core partitions · Jacobi triple product · Sum of squares

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1 Introduction

Three of the most important functions discovered by C. G. J. Jacobi are

$$\vartheta_2(q) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2}, \vartheta_3(q) = \sum_{j=-\infty}^{\infty} q^{j^2} \text{ and}$$
$$\vartheta_4(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2}. \tag{1.1}$$

These functions satisfy the elegant "Pythagorean-type" identity

$$\vartheta_3^4(q) = \vartheta_4^4(q) + \vartheta_2^4(q). \tag{1.2}$$

The above identity can be rewritten as

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+n^2}\right)^2 = \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2}\right)^2 + \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{(m+1/2)^2+(n+1/2)^2}\right)^2. \tag{1.3}$$

One way of proving (1.3) is to first express $\vartheta_j(q)$ in terms of infinite products using the Jacobi triple product identity, namely,

$$\vartheta_2(q) = 2q^{1/4}\prod_{j=1}^{\infty}\frac{(1-q^{4j})^2}{(1-q^{2j})},$$
$$\vartheta_3(q) = \prod_{j=1}^{\infty}\frac{(1-q^{2j})^5}{(1-q^j)^2(1-q^{4j})^2} \text{ and}$$
$$\vartheta_4(q) = \prod_{j=1}^{\infty}\frac{(1-q^j)^2}{(1-q^{2j})}.$$

Identity (1.3) then follows from the identity

$$\frac{\eta^{24}(2\tau)}{\eta^{24}(\tau)} = \frac{\eta^8(4\tau)}{\eta^8(\tau)} + 16\frac{\eta^{16}(4\tau)}{\eta^{16}(\tau)}. \tag{1.4}$$

Here $\eta(\tau)$ is the Dedekind η-function given by

$$\eta(\tau) = e^{2\pi i\tau/24}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}), \quad \operatorname{Im}\tau > 0. \tag{1.5}$$

Identity (1.4) follows from the fact that $\eta^8(4\tau)/\eta^8(\tau)$ is a Hauptmodul for $\Gamma_0(4)$ and $\eta^{24}(2\tau)/\eta^{24}(\tau)$ is invariant under $\Gamma_0(2)$.

In 1991, J. M. Borwein and P. B. Borwein [7, (2.3)] discovered an identity which is amazingly similar to (1.2). Their identity is given by

$$\left(\sum_{\ell,m=-\infty}^{\infty} q^{\ell^2+\ell m+m^2}\right)^3 = \left(\sum_{\ell,m=-\infty}^{\infty} e^{2\pi i(\ell-m)/3} q^{\ell^2+\ell m+m^2}\right)^3 + \left(\sum_{\ell,m=-\infty}^{\infty} q^{\left(\ell+\frac{1}{3}\right)^2+\left(\ell+\frac{1}{3}\right)\left(m+\frac{1}{3}\right)+\left(m+\frac{1}{3}\right)^2}\right)^3. \quad (1.6)$$

A proof of (1.6) in a similar way as in the proof of (1.3) would be to first express the theta series

$$a(q) = \sum_{\ell,m=-\infty}^{\infty} q^{\ell^2+\ell m+m^2},$$

$$b(q) = \sum_{\ell,m=-\infty}^{\infty} e^{2\pi i(\ell-m)/3} q^{\ell^2+\ell m+m^2}$$

and

$$c(q) = \sum_{\ell,m=-\infty}^{\infty} q^{\left(\ell+\frac{1}{3}\right)^2+\left(\ell+\frac{1}{3}\right)\left(m+\frac{1}{3}\right)+\left(m+\frac{1}{3}\right)^2}$$

in terms of infinite products or linear combinations of infinite products. Such representations exist and they are given by

$$a(q) = \prod_{j=1}^{\infty} \frac{(1-q^j)^3}{(1-q^{3j})} + 3q \prod_{j=1}^{\infty} \frac{(1-q^{9j})^3}{(1-q^{3j})}, \quad (1.7)$$

$$b(q) = \prod_{j=1}^{\infty} \frac{(1-q^j)^3}{(1-q^{3j})} \quad (1.8)$$

and

$$c(q) = 3q^{1/3} \prod_{j=1}^{\infty} \frac{(1-q^{3j})^3}{(1-q^j)}. \quad (1.9)$$

Identity (1.6) then follows from the identity

$$1 + 27\frac{\eta^{12}(3\tau)}{\eta^{12}(\tau)} = \left(1 + 9\frac{\eta^3(9\tau)}{\eta^3(\tau)}\right)^3. \quad (1.10)$$

Identity (1.10) follows from the fact that $\eta^3(9\tau)/\eta^3(\tau)$ is a Hauptmodul of $\Gamma_0(9)$ and that $\eta^{12}(3\tau)/\eta^{12}(\tau)$ is a modular function invariant under $\Gamma_0(9)$.

The most elementary proof of (1.8) is due to Borwein et al. [8]. The proof involves Euler's identity

$$(z;q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k z^k q^{k(k-1)/2}}{(q;q)_k},$$

where

$$(a;q)_\infty = \prod_{j=1}^{\infty}(1-aq^j), \quad |q|<1,$$
$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

We reproduce Borweins–Garvan's proof of (1.8) in Sect. 2. To obtain (1.9), they used a transformation formula for the series representations of $b(q)$ and $c(q)$. Although they asserted that (1.9) can also be obtained using Euler's identity, but instead of proving (1.9) using idea similar to the proof of (1.8), they gave a proof which requires transformation formulas satisfied by $\eta(\tau)$ and some identities satisfied by $a(q)$ and $c(q)$.

In this note, we will provide the proof of (1.9) along the same line as Borweins–Garvan's proof of (1.8) and independent of the transformation formula of $\eta(\tau)$. As a key step in this proof, we establish the following representation for $c(q)$:

$$c(q) = 3\sum_{m,n=-\infty}^{\infty} q^{3m^2+3n^2+3mn+m+2n+1}. \tag{1.11}$$

This identity can be found in the work of Ye [12] and the book of Cooper [9, Lemma 3.13]. We shall give two new proofs for it as well as two new representations for $c(q)$ (see Theorem 2).

The product representation (1.9) relates $c(q)$ closely to 3-core partitions. A partition is said to be a t-core if it has no hook numbers that are multiples of t. The number of t-core partitions of n is denoted by $a_t(n)$. Garvan et al. [10] found that

$$\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{j=1}^{\infty} \frac{(1-q^{tj})^t}{1-q^j}. \tag{1.12}$$

The product representation of $c(q)$ given in (1.9) showed that

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{1}{3}q^{-1/3}c(q). \tag{1.13}$$

This identity shows that $a_3(n)$ is closely related to the representations of integers by quadratic forms. Indeed, using Ramanujan's theta function identities and series manipulations, Baruah and Nath [4] proved the following relation:

Theorem 1 *Let $u(n)$ denote the number of representations of a nonnegative integer n in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$. Then*

$$u(12n + 4) = 6a_3(n). \tag{1.14}$$

In one of our proofs for (1.11), we divide the set

$$M(n) = \{(x, y) \in \mathbb{Z}^2 | n = x^2 + 3y^2, 3 \nmid x, x \equiv y \pmod 2\}$$

into six equinumerous subclasses. Moreover, for such subclasses of $M(12n + 4)$, we show that each of them has cardinality $a_3(n)$. This together with the fact that $u(12n + 4) = |M(12n + 4)|$ give a "combinatorial" proof for Theorem 1.

Finally, we also apply the method of Borweins and Garvan to give a proof of Jacobi's triple product identity.

2 Euler's identity and product formula for $c(q)$

In this section, we reproduce the proof of (1.8) given by Borwein et al. We first recall Euler's identity

$$(-z; q)_\infty = \sum_{k=0}^{\infty} \frac{z^k q^{k(k-1)/2}}{(q; q)_k}. \tag{2.1}$$

Observe that if $\omega = e^{2\pi i/3}$ then

$$(-z^3; q^3)_\infty = (-z; q)_\infty (-z\omega; q)_\infty (-z\omega^2; q)_\infty. \tag{2.2}$$

Using (2.1), we deduce from (2.2) that

$$\sum_{k=0}^{\infty} \frac{z^{3k} q^{3k(k-1)/2}}{(q^3; q^3)_k} = \sum_{n_0, n_1, n_2 \geq 0} \frac{\omega^{n_1+2n_2} q^{n_0(n_0-1)/2+n_1(n_1-1)/2+n_2(n_2-1)/2}}{(q; q)_{n_0} (q; q)_{n_1} (q; q)_{n_2}} z^{n_0+n_1+n_2}.$$

Equating the coefficients of z^{3k} on both sides of the above identity yields

$$\frac{q^{3k(k-1)/2}}{(q^3; q^3)_k} = \sum_{n_0+n_1+n_2=3k} \omega^{n_1+2n_2} \frac{q^{n_0(n_0-1)/2+n_1(n_1-1)/2+n_2(n_2-1)/2}}{(q; q)_{n_0} (q; q)_{n_1} (q; q)_{n_2}}. \tag{2.3}$$

Let $m_i = n_i - k$. Observe that the condition $n_0 + n_1 + n_2 = 3k$ is now replaced by $m_0 + m_1 + m_2 = 0$ and

$$\frac{n_0(n_0-1)}{2} + \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} = \frac{m_0^2 + m_1^2 + m_2^2}{2} + \frac{3k(k-1)}{2}.$$

This implies that (2.3) may be written as

$$\frac{1}{(q^3;q^3)_k} = \sum_{\substack{m_0,m_1,m_2 \geq -k \\ m_0+m_1+m_2=0}} \frac{\omega^{m_1-m_2} q^{(m_0^2+m_1^2+m_2^2)/2}}{(q;q)_{m_0+k}(q;q)_{m_1+k}(q;q)_{m_2+k}}.$$

Letting $k \to \infty$, we complete the proof of (1.8).

Next, instead of using (2.2), we start with the identity

$$(-zq;q)_\infty = (-zq;q^3)_\infty(-zq^2;q^3)_\infty(-zq^3;q^3)_\infty. \tag{2.4}$$

Using (2.1), we deduce that

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{z^k q^k q^{k(k-1)/2}}{(q;q)_k} \\ &= \sum_{n_0,n_1,n_2 \geq 0} \frac{z^{n_0+n_1+n_2} q^{n_0+2n_1+3n_2+3(n_0(n_0-1)/2+n_1(n_1-1)/2+n_2(n_2-1)/2)}}{(q^3;q^3)_{n_0}(q^3;q^3)_{n_1}(q^3;q^3)_{n_2}}. \end{aligned} \tag{2.5}$$

Comparing the coefficients of z^k on both sides of (2.5), we conclude that

$$\frac{q^{k(k+1)/2}}{(q;q)_k} = \sum_{\substack{n_0,n_1,n_2 \geq 0 \\ n_0+n_1+n_2=k}} \frac{q^{n_0+2n_1+3n_2+3(n_0(n_0-1)/2+n_1(n_1-1)/2+n_2(n_2-1)/2)}}{(q^3;q^3)_{n_0}(q^3;q^3)_{n_1}(q^3;q^3)_{n_2}}. \tag{2.6}$$

Replacing k by $3k$ and using the substitutions $n_i = m_i + k$, we deduce that

$$\frac{1}{(q;q)_{3k}} = \sum_{\substack{m_0,m_1,m_2 \geq -k \\ m_0+m_1+m_2=0}} \frac{q^{3(m_0^2+m_1^2+m_2^2)/2+3k(m_0+m_1+m_2)/2+(-m_0+m_1+3m_2)/2}}{(q^3;q^3)_{m_0+k}(q^3;q^3)_{m_1+k}(q^3;q^3)_{m_2+k}}.$$

Letting $k \to \infty$, we deduce that

$$\sum_{m_1,m_2=-\infty}^{\infty} q^{3m_1^2+3m_2^2+3m_1m_2+m_1+2m_2} = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

Hence,

$$q^{1/3}\sum_{m,n=-\infty}^{\infty} q^{3m^2+3n^2+3mn+m+2n} = q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

In order to derive (1.9), it suffices to show that

$$c(q^3) = 3\sum_{m,n=-\infty}^{\infty} q^{9m^2+9n^2+9mn+3m+6n+1}. \tag{2.7}$$

We will establish (2.7) in the next section.

3 The product representation of Borweins' $c(q)$

In this section, we present three proofs of (2.7). The first proof of (2.7) we present is due to Ye [12, Lemma 2.4]. Ye discovered (2.7) and used it to show that

$$\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+7n^2} = \frac{1}{3}a(q) - \frac{1}{3}a(q^3) + a(q^9).$$

Ye's proof of (2.7) Note that

$$c(q^3) = \sum_{m,n=-\infty}^{\infty} q^{3(m^2+mn+n^2+m+n)+1}.$$

Now, we split the series into three series according to the congruences

$$n - m \equiv 0, 1, -1 \pmod 3.$$

Suppose $n - m \equiv 0 \pmod 3$. Let $n - m = 3k$. Now we may rearrange the above and obtain

$$n - 2k = m + k = \nu.$$

This gives $n = 2k + \nu$ and $m = \nu - k$. Hence, we deduce that

$$\sum_{\substack{n,m=-\infty \\ n-m\equiv 0 \pmod 3}}^{\infty} q^{3(m^2+mn+n^2+m+n)+1} = \sum_{\nu,k=-\infty}^{\infty} q^{9\nu^2+9\nu k+9k^2+6\nu+3k+1}.$$

For $n - m \equiv 1 \pmod 3$, we write $n - m = 3k + 1$ and introduce ν using the relation

$$n - k = m + 2k + 1 = -\nu,$$

which yields $n = k - \nu$ and $m = -\nu - 2k - 1$. This gives

$$\sum_{\substack{n,m=-\infty \\ n-m\equiv 1 \pmod 3}}^{\infty} q^{3(m^2+mn+n^2+m+n)+1} = \sum_{\nu,k=-\infty}^{\infty} q^{9\nu^2+9\nu k+9k^2+6k+3\nu+1}.$$

The last case follows from the fact that

$$\sum_{\substack{n,m=-\infty \\ n-m\equiv 1 \pmod 3}}^{\infty} q^{3(m^2+mn+n^2+m+n)+1} = \sum_{\substack{n,m=-\infty \\ m-n\equiv -1 \pmod 3}}^{\infty} q^{3(m^2+mn+n^2+m+n)+1}.$$

Collecting our observations, we find that

$$c(q^3) = 3 \sum_{\nu,k=-\infty}^{\infty} q^{9\nu^2+9\nu k+9k^2+6\nu+3k+1}.$$

□

We now give another proof of (2.7). This proof is longer but we obtain the following identities involving $c(q^3)$:

$$c(q^3) = \sum_{\substack{h,k=-\infty \\ h+2k\equiv 2 \pmod 3}}^{\infty} q^{h^2+hk+k^2} \tag{3.1}$$

$$= 3 \sum_{\substack{m,n=-\infty \\ m\equiv 1 \pmod 3 \\ n\equiv 0 \pmod 3}}^{\infty} q^{N(m+\omega n)} \tag{3.2}$$

$$= 3 \sum_{m,n=-\infty}^{\infty} q^{9m^2+9n^2+9mn+3m+6n+1}, \tag{3.3}$$

where $\omega = e^{\pi i/3}$ and

$$N(a + b\omega) = a^2 + ab + b^2. \tag{3.4}$$

Second proof of (2.7) Let

$$A = \{(m, n)|m, n \in \mathbb{Z}\},$$

and

$$B = \{(x, y)|x, y \in \mathbb{Z}, x \equiv y \pmod 2\}.$$

The maps $f : A \to B$ and $g : B \to A$ defined by

$$f(m,n) = (2m+n, n) \text{ and } g(x,y) = ((x-y)/2, y), \tag{3.5}$$

set up a one to one correspondence between A and B.

Now,

$$\begin{aligned} q^{-4/3}c(q^4) &= \sum_{m,n=-\infty}^{\infty} q^{(2m+n)^2+3n^2+4(m+n)} \\ &= \sum_{\substack{x,y=-\infty \\ x\equiv y \pmod 2}}^{\infty} q^{x^2+3y^2+2(x+y)} := L(q). \end{aligned}$$

Hence,

$$\begin{aligned} c(q^{12}) = q^4 L(q^3) &= \sum_{\substack{x,y=-\infty \\ x\equiv y \pmod 2}}^{\infty} q^{3(x+1)^2+(3y+1)^2} = \sum_{\substack{s,t=-\infty \\ s\equiv t \pmod 2}}^{\infty} q^{3s^2+(3t+2)^2} \\ &= \sum_{\substack{\nu,\mu=-\infty \\ \nu\equiv\mu \pmod 2 \\ \mu\equiv 2 \pmod 3}}^{\infty} q^{3\nu^2+\mu^2} = \sum_{\substack{h,k=-\infty \\ h+2k\equiv 2 \pmod 3}}^{\infty} q^{4(h^2+hk+k^2)}, \end{aligned} \tag{3.6}$$

where we have used (3.5) in the last equality. This immediately yields a different representation of $c(q^3)$, namely,

$$c(q^3) = \sum_{\substack{h,k=-\infty \\ h+2k\equiv 2 \pmod 3}}^{\infty} q^{h^2+hk+k^2}.$$

Now, the condition $x + 2y \equiv 2 \pmod 3$ if and only if

$$(x, y) \equiv (0,1), (1,2) \text{ or } (2,0) \pmod 3.$$

Hence,

$$\sum_{\substack{h,k=-\infty \\ h+2k\equiv 2 \pmod 3}}^{\infty} q^{h^2+hk+k^2} = S_1 + S_2 + S_3, \tag{3.7}$$

where

$$S_1 = \sum_{\substack{h,k=-\infty \\ (h,k)\equiv(0,1) \pmod 3}}^{\infty} q^{h^2+hk+k^2},$$

$$S_2 = \sum_{\substack{h,k=-\infty \\ (h,k)\equiv(1,2) \pmod 3}}^{\infty} q^{h^2+hk+k^2}$$

and

$$S_3 = \sum_{\substack{h,k=-\infty \\ (h,k)\equiv(2,0) \pmod 3}}^{\infty} q^{h^2+hk+k^2}.$$

Now, with $s = -h$ and $t = -k$, we find that

$$S_1 = \sum_{\substack{h,k=-\infty \\ (h,k)\equiv(0,1) \pmod 3}}^{\infty} q^{h^2+hk+k^2} = \sum_{\substack{s,t=-\infty \\ (s,t)\equiv(0,-1) \pmod 3}}^{\infty} q^{s^2+st+t^2} = S_3.$$

To complete the proof of (3.2), it remains to show that $S_1 = S_2$.

Note that ω is a sixth root of unity and satisfies $\omega^2 - \omega + 1 = 0$. We have $N(\omega^2) = N(\omega - 1) = 1$. Using the definition of $N(\cdot)$ given in (3.4), we deduce that

$$\begin{aligned} S_2 &= \sum_{\substack{m,n=-\infty \\ (m,n)\equiv(1,2) \pmod 3}}^{\infty} q^{m^2+mn+n^2} = \sum_{\substack{m,n=-\infty \\ (m,n)\equiv(1,2) \pmod 3}}^{\infty} q^{N(m+\omega n)} \\ &= \sum_{\substack{m,n=-\infty \\ (m,n)\equiv(1,2) \pmod 3}}^{\infty} q^{N(\omega^2(m+\omega n))} = \sum_{\substack{m,n=-\infty \\ (m,n)\equiv(1,2) \pmod 3}}^{\infty} q^{N(-m-n+\omega m)} \\ &= \sum_{\substack{m,n=-\infty \\ (m,n)\equiv(0,1) \pmod 3}}^{\infty} q^{N(m+\omega n)} = S_1. \end{aligned}$$

Hence,

$$c(q^3) = 3 \sum_{\substack{m,n=-\infty \\ m\equiv 1 \pmod 3 \\ n\equiv 0 \pmod 3}}^{\infty} q^{N(m+\omega n)} = 3 \sum_{m,n=-\infty}^{\infty} q^{9m^2+9n^2+9mn+6m+3n+1}.$$

□

To give the third proof of (2.7), we first establish the following result, which is of independent interest.

Lemma 1 *Let*

$$\begin{aligned}
M(n) &= \{(x, y) \in \mathbb{Z}^2 | n = x^2 + 3y^2, 3 \nmid x, x \equiv y \pmod 2\},\\
A_1(n) &= \{(x, y) \in M(n) | (x, y) \equiv (1, 0) \pmod 3\},\\
A_2(n) &= \{(x, y) \in M(n) | (x, y) \equiv (1, 1) \pmod 3\},\\
A_3(n) &= \{(x, y) \in M(n) | (x, y) \equiv (1, 2) \pmod 3\},\\
A_4(n) &= \{(x, y) \in M(n) | (x, y) \equiv (2, 0) \pmod 3\},\\
A_5(n) &= \{(x, y) \in M(n) | (x, y) \equiv (2, 1) \pmod 3\},\\
A_6(n) &= \{(x, y) \in M(n) | (x, y) \equiv (2, 2) \pmod 3\}.
\end{aligned}$$

Then

$$|A_1(n)| = |A_2(n)| = |A_3(n)| = |A_4(n)| = |A_5(n)| = |A_6(n)| = \frac{1}{6}|M(n)|. \quad (3.8)$$

Proof It is immediate that

$$M(n) = \bigcup_{i=1}^{6} A_i(n).$$

Using the bijection $(x, y) \to (-x, -y)$ we deduce that

$$|A_1(n)| = |A_4(n)|, \quad |A_2(n)| = |A_6(n)|, \quad |A_3(n)| = |A_5(n)|. \quad (3.9)$$

Using the bijection $(x, y) \to (x, -y)$ we get

$$|A_2(n)| = |A_3(n)|. \quad (3.10)$$

It remains to show that

$$|A_3(n)| = |A_4(n)|. \quad (3.11)$$

Note that

$$\left(\frac{x - 3y}{2}\right)^2 + 3\left(\frac{x + y}{2}\right)^2 = x^2 + 3y^2. \quad (3.12)$$

It is not difficult to see that $(x, y) \to (\frac{x-3y}{2}, \frac{x+y}{2})$ gives a bijection between $A_3(n)$ and $A_4(n)$. Thus (3.11) holds and the lemma is proved. □

We now use Lemma 1 to deduce the following:

Theorem 2 *The following identities hold:*

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}$$
$$= 3\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{3m^2+3mn+3n^2+m+2n} \tag{3.13}$$
$$= 3\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{3m^2+3mn+3n^2+2m+4n+1} \tag{3.14}$$
$$= 3\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{3m^2+3mn+3n^2+4m+5n+2}. \tag{3.15}$$

Proof Let

$$H(q) := q^{-1/3}c(q) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}. \tag{3.16}$$

Replacing q by q^4, we find that

$$H(q^4) = \sum_{m,n=-\infty}^{\infty} q^{(m+2n)^2+3m^2+4(m+n)}$$
$$= \sum_{\substack{s,t=-\infty\\ s\equiv t \pmod 2}}^{\infty} q^{3s^2+t^2+2(s+t)}$$
$$= \sum_{\substack{s,t=-\infty\\ s\equiv t \pmod 2}}^{\infty} q^{3s^2+2s-1+(t+1)^2}, \tag{3.17}$$

where we have used the bijection (3.5) in the second equality.

Hence,

$$q^4H(q^{12}) = \sum_{\substack{s,t=-\infty\\ s\equiv t \pmod 2}}^{\infty} q^{(3s+1)^2+3(t+1)^2} \quad (x = 3s+1,\ y = t+1)$$
$$= \sum_{\substack{x,y=-\infty\\ x\equiv y \pmod 2\\ x\equiv 1 \pmod 3}}^{\infty} q^{x^2+3y^2}. \tag{3.18}$$

By Lemma 1, we deduce that

$$q^4 H(q^{12}) = 3 \sum_{\substack{x,y=-\infty \\ x\equiv y \pmod 2 \\ (x,y)\equiv(2,0) \pmod 3}}^{\infty} q^{x^2+3y^2}. \tag{3.19}$$

Now we substitute $(x, y) = (3m + 6n + 2, 3m)$ in (3.19). It follows that

$$q^4 H(q^{12}) = 3q^4 \sum_{m,n=-\infty}^{\infty} q^{12(3m^2+3mn+3n^2+m+2n)}. \tag{3.20}$$

This proves (3.13).

Similarly, using Lemma 1 and (3.18) we deduce that

$$q^4 H(q^{12}) = 3 \sum_{\substack{x,y=-\infty \\ x\equiv y \pmod 2 \\ (x,y)\equiv(2,2) \pmod 3}}^{\infty} q^{x^2+3y^2}. \tag{3.21}$$

Making the substitution $(x, y) = (3m + 6n + 2, 3m + 2)$ we arrive at (3.14).

In the similar way, using Lemma 1 and (3.18) we deduce that

$$q^4 H(q^{12}) = 3 \sum_{\substack{x,y=-\infty \\ x\equiv y \pmod 2 \\ (x,y)\equiv(1,2) \pmod 3}}^{\infty} q^{x^2+3y^2}. \tag{3.22}$$

Making the substitution $(x, y) = (3m + 6n + 4, 3m + 2)$ we deduce (3.15). □

Remark 1 The identity (3.13) proves (2.7). The identities (3.13)–(3.15) are equivalent to $S_1 = S_2 = S_3$ where S_1, S_2, S_3 are given in (3.7). Identity (3.18) is the same as (3.6).

We now present a new proof of Theorem 1 using Lemma 1.

Proof of Theorem 1 From (1.13) we find that

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{1}{3}H(q). \tag{3.23}$$

From (3.19), we deduce that

$$a_3(n) = |A_4(12n + 4)|. \tag{3.24}$$

We note that $12n+4=x^2+3y^2$ implies that $x \not\equiv 0 \pmod 3$ and $x \equiv y \pmod 2$. Therefore, we find that

$$u(12n+4)=|M(12n+4)|.$$

Hence by Lemma 1, we conclude that

$$u(12n+4)=6|A_4(12n+4)|. \tag{3.25}$$

Combining this with (3.24), we deduce that $u(12n+4)=6a_3(n)$. □

4 A proof of the Jacobi triple product identity using elliptic functions

One of the several expressions of the Jacobi triple product is given by

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2} e^{2iuj} = \prod_{j=1}^{\infty}(1-q^{2j})(1-e^{2iu}q^{2j-1})(1-e^{-2iu}q^{2j-1}). \tag{4.1}$$

In [5, p. 42, Theorem 6], R. Bellman stated the above identity and remarked that "There are no simple proofs known of the complete result, but there are a number of fairly straightforward ways of deriving the partial result

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2} e^{2iuj} = G(q)\prod_{j=1}^{\infty}(1-e^{2iu}q^{2j-1})(1-e^{-2iu}q^{2j-1}), \tag{4.2}$$

where $G(q)$ is independent of u". In 1965, motivated by Bellman's remark, G.E. Andrews [1] gave an elementary proof of (4.1). Andrews' proof, unlike the proofs that Bellman had in mind, does not involve (4.2) as an intermediate step.

There are now many proofs of (4.1) and most of these proofs involve first deriving (4.2), followed by determining $G(q)$. For examples of such proofs, see [11, Theorem 352] and [6, Theorem 3.1]. In this section, we illustrate an example of such proofs that involve elliptic functions.

Let $q=e^{\pi i\tau}$ with $\operatorname{Im}\tau>0$ throughout this section. The left-hand side of (4.1) is usually denoted by

$$\vartheta_4(u|\tau) := \sum_{j=-\infty}^{\infty}(-1)^j q^{j^2} e^{2iuj}.$$

Let

$$P(u|\tau)=\prod_{j=1}^{\infty}(1-e^{2iu}q^{2j-1})(1-e^{-2iu}q^{2j-1}),$$

and observe that both $\vartheta_4(u|\tau)$ and $P(u|\tau)$ satisfy the functional equations

$$F(u+\pi|\tau) = F(u|\tau) \quad \text{and} \quad F(u+\pi\tau|\tau) = -q^{-1}e^{-2iu}F(u|\tau).$$

This implies that $\vartheta_4(u|\tau)/P(u|\tau)$ is an elliptic function with periods π and $\pi\tau$ with no pole. By Liouville's Theorem, we deduce that there exists a function $G(q)$ for which (4.2) holds.

It now remains to determine $G(q)$. Using the identity

$$(z;q)_\infty(-z;q)_\infty = (z^2;q^2)_\infty,$$

and (2.1), we find that

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j \frac{q^{k(k-1)/2}q^{j(j-1)/2}}{(q;q)_k(q;q)_j} z^{k+j} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell q^{\ell(\ell-1)}}{(q^2;q^2)_\ell}z^{2\ell}.$$

We write the above identity as

$$\begin{aligned}\textstyle\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^j q^{k(k-1)/2}q^{j(j-1)/2}\frac{(q^{k+1};q)_\infty(q^{j+1};q)_\infty}{(q;q)_\infty(q;q)_\infty}z^{k+j}\\ = \textstyle\sum_{\ell=0}^{\infty}\frac{(-1)^\ell q^{\ell(\ell-1)}}{(q^2;q^2)_\ell}z^{2\ell}.\end{aligned}$$

Comparing the coefficients of $z^{2\ell}$ on both sides of the above and replacing k and j by $\ell+t$ and $\ell+s$, respectively, we deduce that

$$\sum_{\substack{s,t=-\infty\\ s+t=0}}^{\infty}(-1)^s q^{s^2}\frac{(q^{\ell+t+1};q)_\infty(q^{\ell+s+1};q)_\infty}{(q;q)_\infty(q;q)_\infty} = \frac{1}{(q^2;q^2)_\ell},$$

or

$$\sum_{s=-\ell}^{\ell}\frac{(-1)^s q^{s^2}}{(q;q)_{s+\ell}(q;q)_{\ell-s}} = \frac{1}{(q^2;q^2)_\ell}, \tag{4.3}$$

since $(q^{\ell-s+1};q)_\infty(q^{\ell+s+1};q)_\infty = 0$ for $s < -\ell$ and $s > \ell$. Letting $\ell \to \infty$ yields

$$\sum_{s=-\infty}^{\infty}(-1)^s q^{s^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \tag{4.4}$$

The left-hand side of (4.4) is the left-hand side of (4.2) with $u=0$. Comparing the right-hand side of (4.4) and (4.2) with $u=0$ yields

$$G(q) = (q^2;q^2)_\infty. \tag{4.5}$$

Remark 2 G.E. Andrews informed us that (4.3) follows from the finite form of the Jacobi triple product identity [2, p. 49, Example 1]. He shared with us the evaluation of $G(q)$ (see (4.5)) he liked best, which is the one given in [3, Section 3], where the Frobenius symbol for partitions arises naturally.

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