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The Fundamental Theorem of Arithmetic and q-series

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In this article, we show a connection between a polynomial equation of k variables and the Fundamental Theorem of Arithmetic. We derive an identity from this observation and give a proof of one of L. Euler's famous identities. We then connect this identity to identities discovered by N.J. Fine.

The fundamental theorem of arithmetic and a simple identity

The fundamental theorem of arithmetic states that if $n > 1$ is a positive integer then n can be written uniquely in the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \tag{1}$$

where $p_1 < p_2 < \cdots < p_k$ are primes. This representation allows us to derive the number of distinct positive divisors of n in two ways. The first way is to observe that any positive divisor d of n must be of the form

$$d = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k},$$

where $0 \le b_j \le \alpha_j$ for $1 \le j \le k$. The number of distinct k-tuples $(b_1, b_2, \cdots, b_k)$ is therefore

$$(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k).$$

Since each k-tuple corresponds to a positive divisor of n, we conclude that the number of distinct positive divisors of n is

$$(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k).$$

Next, we count the number of distinct positive divisors in another way. First, we note that 1 is a divisor. Next, the number of divisors of n with largest prime factor p_1

is α_1. The number of divisors with largest prime factor p_2 is $(1+\alpha_1)\alpha_2$. In general, the number of divisors of n with largest prime factor p_s is

$$(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{s-1})\alpha_s.$$

In other words, the number of divisors of n is also given by

$$1+\alpha_1+(1+\alpha_1)\alpha_2+\cdots+(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{k-1})\alpha_k.$$

This yields the identity

$$\begin{aligned}1+\alpha_1+(1+\alpha_1)\alpha_2+\cdots+(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{k-1})\alpha_k\\ =(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k).\end{aligned} \tag{2}$$

The identity (2) motivates us to ask if the relation holds with α_j replaced by independent variables x_j. The answer is affirmative and can be proved using mathematical induction. It is clear that $1+x_1+(1+x_1)x_2=(1+x_1)(1+x_2)$. Also, suppose

$$\begin{aligned}1+x_2+(1+x_2)x_3+\cdots+(1+x_2)(1+x_3)\cdots(1+x_{k-1})x_k\\ =(1+x_2)(1+x_3)\cdots(1+x_k),\end{aligned}$$

multiplying both sides by $(1+x_1)$ yields

$$\begin{aligned}1+x_1+(1+x_1)x_2+(1+x_1)(1+x_2)x_3+\cdots\\ +(1+x_1)(1+x_2)(1+x_3)\cdots(1+x_{k-1})x_k\\ =(1+x_1)(1+x_2)(1+x_3)\cdots(1+x_k)\end{aligned} \tag{3}$$

and completes the proof of the identity using mathematical induction.

Identity (3) is due to Euler [**5**]. The proof above is essentially his, and we will soon see his application. If we set

$$x_i=\frac{-z}{z+z_i}\quad\text{for each } i\text{, so that}\quad 1+x_i=\frac{z_i}{z+z_i},$$

and divide both sides by z, then it becomes an even older identity due to F. Nicole [**12**], which also appears in the beautiful paper [**11**]. If we instead set

$$x_i=\frac{-a_i}{1+a_i}\quad\text{for each } i\text{, so that}\quad 1+x_i=\frac{1}{1+a_i},$$

then it becomes problem 101 in [**10**]. The variation $1+a_i=c_i$ is in [**9**], and another form will be observed in our concluding section. Problem 104 in [**10**] comes from setting

$$x_i=\frac{z-b_i}{b_i}\quad\text{for each } i\text{, so that}\quad 1+x_i=\frac{z}{b_i}.$$

Finally, if we replace x_i by $-y_i$ in (3), we get problem 5 on the 1952 Putnam Exam [**7**]. Because of the unrestricted nature of the variables and the connection with the fundamental theorem of arithmetic, which is due to the second author and is apparently new, we believe that (3) is the essential form of this fact. Some of the references above were previously collected in [**8**, p. 115, Exercises 18-22].

An identity of Euler

Let $q, a \in \mathbf{C}$ with $|q| < 1$. Let n be a non-negative integer and define

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=1}^{n}(1 - aq^{j-1}) & \text{otherwise.} \end{cases}$$

We also set

$$(a;q)_\infty = \lim_{n\to\infty} \prod_{j=1}^{n}(1 - aq^{j-1}).$$

We now let $x_j = \dfrac{tq^{j-1}}{1 - tq^{j-1}}$ in (3), where $t, q \in \mathbf{C}$. Note that for $1 \le j \le k$,

$$1 + x_j = \frac{1}{1 - tq^{j-1}}.$$

Identity (3) becomes

$$1 + t\sum_{j=1}^{k} \frac{q^{j-1}}{(t;q)_j} = \frac{1}{(t;q)_k}. \tag{4}$$

If we let $k \to \infty$ in (4), then we obtain the identity

$$1 + t\sum_{j=1}^{\infty} \frac{q^{j-1}}{(t;q)_j} = \frac{1}{(t;q)_\infty}. \tag{5}$$

One should compare (5) with the well-known identity

$$1 + \sum_{j=1}^{\infty} \frac{t^j}{(q;q)_j} = \frac{1}{(t;q)_\infty} \tag{6}$$

of Euler [**4**]; see also [**8**, Section 3.5], for example.

If we replace t by tq in (5) and set $t = 1$, we arrive at the following well-known identity of Euler:

$$1 + \sum_{j=1}^{\infty} \frac{q^j}{(q;q)_j} = \frac{1}{(q;q)_\infty}.$$

Note that the above identity also follows by setting $t = q$ in (6).

Identity (4) is a specialization of an identity found in [**8**, p. 115, Exercise 23], namely,

$$\frac{(ax;q)_n}{(bx;q)_n} = 1 + x(b-a)\sum_{j=1}^{n} \frac{(ax;q)_{j-1}}{(bx;q)_j} q^{j-1}. \tag{7}$$

The proof of (7) is similar to the proof of (4).

Cauchy's identity and Fine's function

Although (6) is a well-known identity of Euler, (5) is hard to locate in the literature. Since the right-hand side of both identities are the same, it is natural to ask whether (5) is equivalent to (6). The answer is affirmative and we will present two proofs establishing the equivalence of these two identities.

The first proof begins with an identity of Cauchy ([**2**]; see also for example [**8**, p. 123]), namely,

$$\frac{1}{(x;q)_{n+1}} = \sum_{j=0}^{\infty} \frac{(q;q)_{n+j}}{(q;q)_j (q;q)_n} x^j, \tag{8}$$

where $|q| < 1$ and $|x| < 1$. Observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{tq^n}{(t;q)_{n+1}} &= \sum_{n=0}^{\infty} tq^n \sum_{j=0}^{\infty} \frac{(q;q)_{n+j}}{(q;q)_j (q;q)_n} t^j \\ &= \sum_{j=0}^{\infty} t^{j+1} \sum_{n=0}^{\infty} \frac{(q;q)_{n+j}}{(q;q)_j (q;q)_n} q^n \\ &= \sum_{j=0}^{\infty} \frac{t^{j+1}}{(q;q)_{j+1}},\end{aligned}$$

where Cauchy's identity (8) is used twice. This shows that the left hand sides of (5) and (6) are equal and so, these two identities are equivalent.

The second proof that (5) and (6) are equivalent follows from two identities discovered by N.J. Fine. Let

$$F(a,b;t) = 1 + \sum_{j=1}^{\infty} \frac{(aq;q)_j}{(bq;q)_j} t^j.$$

In [**6**], Fine established many identities associated with $F(a,b;t;q)$. Two of these identities are [**6**, p. 3, (4.3)]

$$F(a,b;t) = \frac{1}{1-t} + \frac{(b-a)tq}{(1-bq)(1-t)} F(a,bq;tq), \tag{9}$$

and [**6**, p. 5, (6.3)]

$$F(a,b;t) = \frac{1-b}{1-t} F(at/b,t;b). \tag{10}$$

To show that (5) is equivalent to (6), it suffices to show that

$$F(0,1;t) = 1 + \frac{t}{1-t} F(0,t;q). \tag{11}$$

Using (9), we observe that

$$F(0,1;t) = \frac{1}{1-t} + \frac{tq}{(1-q)(1-t)} F(0,q;tq). \tag{12}$$

By (10), we find that

$$F(0,q;tq)=\frac{1-q}{1-tq}F(0,tq;q). \tag{13}$$

Therefore, from (12) and (13), we conclude that

$$\begin{aligned} F(0,1;t) &= \frac{1}{1-t}+\frac{tq}{(1-t)(1-tq)}F(0,tq;q) \\ &= \frac{1}{1-t}+\frac{t}{1-t}\sum_{j=0}^{\infty}\frac{q^{j+1}}{(tq;q)_{j+1}} \\ &= \frac{1}{1-t}+\frac{t}{1-t}\left(F(0,t;q)-1\right) \\ &= 1+\frac{t}{1-t}F(0,t;q), \end{aligned}$$

which is (11).

Fine's identities

If we replace q by q^2 in (5) and let $t=q$, then

$$\frac{1}{(q;q^2)_\infty}=1+\sum_{j=1}^{\infty}\frac{q^{2j-1}}{(q;q^2)_j}. \tag{14}$$

Next, let $x_j=q^j$ in (3) to deduce that

$$(-q;q)_k=1+\sum_{j=1}^{k}(-q;q)_{j-1}q^j. \tag{15}$$

Now, let $k\to\infty$ in (15) to deduce that

$$(-q;q)_\infty=1+\sum_{j=1}^{\infty}(-q;q)_{j-1}q^j. \tag{16}$$

This is essentially Euler's application of (3), and the starting point of his rather cumbersome proof of his pentagonal number identity [**5**]; see also the next section and [**8**, Section 4.1]. Another famous result of Euler ([**4**]; see also [**8**, Section 3.2]) is

$$(-q;q)_\infty=\frac{1}{(q;q^2)_\infty}. \tag{17}$$

Therefore, from (14), (16) and (17), we deduce that

$$1+\sum_{j=1}^{\infty}\frac{q^{2j-1}}{(q;q^2)_j}=1+\sum_{j=1}^{\infty}(-q;q)_{j-1}q^j. \tag{18}$$

In [**6**, (23.9)], Fine gave a generalization of (18), namely,

$$\sum_{m=0}^{\infty}\frac{q^{2m+1}t^{m+1}}{(tq;q^2)_{m+1}}=\sum_{j=0}^{\infty}(q;q)_jq^{j+1}t^{j+1}. \tag{19}$$

An elementary proof of (19) using the theory of partitions was given by Chen and Huang [**3**]. Chen and Huang mentioned (18) and deduced (17) by appealing to an infinite version of (3), namely,

$$\prod_{j=1}^{\infty}(1+f_j(q)) = 1+\sum_{m=1}^{\infty}\prod_{s=1}^{m-1}(1+f_s(q))f_m(q), \tag{20}$$

thereby giving another application of (3).

In the same work, Chen and Huang also discovered that [**3**, Theorem 2.3]

$$q^{2\ell}\sum_{m=1}^{\infty}\frac{t^m q^{(2k+1)m}}{(tq^{2\ell};q^2)_m} = q^{2k+1}\sum_{m=1}^{\infty}\frac{t^m q^{2\ell m}}{(tq^{2k+1};q^2)_m}. \tag{21}$$

We take this opportunity to derive a generalization of (21). Let $a=0$ in (10) and deduce that

$$\frac{1}{1-b}F(0,b;t) = \frac{1}{1-t}F(0,t;b).$$

This yields

$$\frac{1}{t}\left(\sum_{m=1}^{\infty}\frac{t^m}{(b;q)_{m+1}}\right) = \frac{1}{b}\left(\sum_{m=1}^{\infty}\frac{b^m}{(t;q)_{m+1}}\right).$$

Replacing q by q^2, t by tq^{2k+1}, b by $tq^{2\ell}$, we complete the proof of (21).

Euler's pentagonal number identity

In [**1**], Andrews defined

$$f(x,q) = 1-\sum_{j=1}^{\infty}(1-xq)(1-xq^2)\cdots(1-xq^{j-1})x^{j-1}q^j$$

and showed that

$$f(x,q) = 1-x^2q-x^3q^2f(xq,q). \tag{22}$$

From (22), he deduced that

$$f(x,q) = 1+\sum_{m=1}^{\infty}(-1)^m\big(x^{3m-1}q^{m(3m-1)/2}+x^{3m}q^{m(3m+1)/2}\big). \tag{23}$$

Andrews then set $x=1$ in (23) and deduced using (20) that

$$f(1,q) = (q;q)_\infty. \tag{24}$$

Euler's pentagonal number identity

$$(q;q)_\infty = \sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2} \tag{25}$$

then followed from (24) and (23).

Apéry's proof of the irrationality of $\zeta(3)$

In this final section, we connect (3) to Apéry's proof of the irrationality of $\zeta(3)$. In [**13**], A. Van Der Poorten told the story of R. Apéry announcing his proof that the constant

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

is irrational. Apparently, Apéry gave his presentations with a sequence of unlikely assertions, the first one being related to the identity

$$\sum_{k=1}^{K} \frac{a_1 a_2 \cdots a_{k-1}}{(x+a_1)(x+a_2)\cdots(x+a_k)} = \frac{1}{x} - \frac{a_1 a_2 \cdots a_K}{x(x+a_1)(x+a_2)\cdots(x+a_K)}. \tag{26}$$

As illustrated by H. Cohen [**13**, Section 3] with clever substitutions, (26) implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}. \tag{27}$$

It turns out that (26) is a consequence of (3) by simply letting

$$x_j = -\frac{x}{a_j + x}.$$

It is not clear why Apéry included (27) in his list of claims when he presented his work on $\zeta(3)$. The proof which was eventually presented by Poorten in his article did not use (27). But still this identity is interesting in its own right.

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Summary. We connect the fundamental theorem of arithmetic to the Euler-Nichole identity and discuss various useful q-identities.

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